# Convexity

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## Assignment Sheet 6 - Solutions

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**Exercise 1**
Let $P$ be a polyhedron and $F$ a facet. Show $\dim(F) = \dim(P) - 1$.

**Solution:**
Let $P$ be described by the non-redundant system $Ax \leq b$ and $F$ be a facet of $P$. Let $A^{=}x \leq b^{=}$ the system of implicit equalities, $a^T x \leq \beta$ the inequality corresponding to $F$ (by the theorem of the lecture), and $A'x \leq b'$ be the rest of the system.

In the proof of the theorem linking the facets to the inequalities, we saw that there is a point $x^\star \in P$ with

$$\begin{pmatrix} A^{=} \\ a^T \\ A' \end{pmatrix} x \quad \begin{matrix} = \\ = \\ \leq \end{matrix} \quad \begin{pmatrix} b^{=} \\ \beta \\ b' \end{pmatrix}.$$

As $F = \{x \in P : \ a^T x = \beta\}$ and all inequalities in $A'x \leq b'$ are no implicit equalities, the system of implicit equalities of $F$ is exactly one larger than of $P$. Once we have shown that the rank also grew by one, we are done using the characterization of the dimension of a polyhedron. But if the vector $a$ was linearly dependent of the rows in $A^{=}$, the equality $a^T x = \beta$ would be fulfilled by all elements in $P$, hence $F$ would not be a facet.

**Exercise 2**
Prove the Birkhoff- von Neumann Theorem:
The extreme points of the set of doubly stochastic matrices

$$M = \left\{ A = (a_{i,j})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}_{\geq 0} \mid \sum_{i=1}^{n} a_{i,j_0} = \sum_{j=1}^{n} a_{i_0,j} = 1 \ \forall 1 \leq i_0, j_0 \leq n \right\}$$

are precisely the permutation matrices, i.e. $M \cap \{0,1\}^{n \times n}$.

**Solution:**
Note that $M$ is a polytope with constraints

$$\sum_{i=1}^{n} a_{ij} = 1 \qquad \text{for } j = 1, \dots, n \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad \text{for } i = 1, \dots, n$$

$$a_{ij} \geq 0 \qquad \text{for } i, j = 1, \dots, n \tag{2}$$

for the contained matrices $A = (a_{ij})_{i,j}$. Each permutation matrix is extreme, as otherwise at least one 1-entry would be a convex combination of some numbers in $[0,1)$, which is not possible. If we can show that each extreme point of $P$ is a convex combination of permutation matrices, we are basically done.

We will proceed inductively. For $n = 1$, the scalar 1 is the only doubly-stochastic matrix, and a permutation matrix. Let $A$ be a vertex of the polytope described by (1). Hence, $A$ has to fulfil $n^2$ linearly independent inequalities with equality. (Recall the characterization of extreme points.) As the first $2n$ inequalities are linearly independent, at least $n^2 - 2n + 1$ of the other inequalities have to be fulfilled with equality, implying that at least $n^2 - 2n + 1$ of the entries in $A$ are zero. By the pidgeon hole principle, at least one row of $A$ has to consist of $n - 1$ zeroes and one entry equal to 1, say w.l.o.g. $a_{11} = 1$. But then the first column contains $n - 1$ zeroes as well and the matrix $A'$ arising from $A$ by deleting the first row and column is doubly stochastic and of smaller dimension, hence it is the convex combination of permutation matrices.

**Exercise 3 [$\star$]**
Using the separation theorem, prove that the system $Ax = b$, $x \geq 0$ has no solution if and only if there is a vector $c$ s.th. $c^T A \geq 0$ and $c^T b < 0$.

**Solution:**
Assume the system has no solution. All $b'$ for which there exists an $x \geq 0$ with $Ax = b'$ are in the cone generated by the columns of $A$, call this cone $S$. As $b \notin S$, there exists a separating hyperplane $\{x : \ c^T x = \gamma\}$ with $c^T b < \gamma$ and $c^T y \geq \gamma$ for any $y \in S$. As $S$ is a cone, we can replace *gamma* by 0 (in particular, $\gamma \leq 0$). Hence, $c$ is the desired vector.
On the other hand, if there exists such a vector it can be interpreted as a separating hyperplane. Or, to give it another angle, if $c^T x = 0$ is the hyperplane and there was a solution $x^\star$, $0 > c^T b = c^T A x^\star \geq 0$ yields a contradiction.

**Exercise 4**
Let $F$ be an (inclusion-wise) minimal face of a polyhedron $P = \{x : \ Ax \leq b\}$. Show that the following holds.

$$\forall x, y \in F : \ Ax = Ay.$$

**Solution:**
Let $A'x \leq b'$ be a maximum subsystem s.th. for all $x, y \in F$ we have $A'x = A'y$ and let $A''x \leq b''$ be the remaining system. If the remaining system is empty, we are done.
Otherwise, we want to find a proper face of $F$. By transitivity, this will be a face of $P$, contradicting minimality.
Pick $x, y \in F$ s.th. there is at least one inequality with $a^T x \neq a^T y$ in the system $A''x \leq b''$. Consider $z(\lambda) := x + \lambda(y - x)$. First note that $A'z(\lambda) = A'x$ for any $\lambda$. As $z(0)$ is in $F$ and due to the choice of $x, y$, there is a $\lambda_0$ with $z(\lambda)$ being feasible for $A''x \leq b''$ and one of the inequalities is fulfilled with equality, say $a_1^T z(\lambda_0) = \beta_1$. (Metaphorically speaking, we start at $x$ and, by altering $\lambda$, walk in the direction of $y$ or straight away from $y$ for negative $\lambda$ until we hit the boundary of $F$.) But then $F \cap \{x : \ a_1^T x = \beta_1\}$ is a non-empty intersection of $F$ with a supporting hyperplane, yielding the contradiction.