Convexity

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Assignment Sheet 4 - Solutions

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Exercise 1

Consider $\Lambda = \mathbb{Z}^n$. Show that there exists a convex body K not containing any integer point in its interior such that for each nonzero integral vector y one has

$$\max_{x \in K} y^T x - \min_{x \in K} y^T x \geq n.$$

(For simplicity, it might contain lattice points on its boundary.)

Solution:

Consider the standard n-simplex $S := \{x : 0 \leq x \leq n, \|x\|_1 \leq n\} \subseteq \mathbb{R}^n$. Obviously, S is a convex body.

First observe that

$$p \in S \cap \mathbb{Z}^n \quad \Rightarrow \quad p \in \partial S,$$

i.e. any integral point $p \in S$ is on its boundary: whenever there is some $p_i = 0$, one of the inequalities $0 \leq p_i$ is tight. Hence, an integral point in the interior of S fulfils $p_i \geq 1$. But then $p_i = 1$ for all i and the inequality $\|p\|_1 \leq n$ is tight. Thus there is no integral point in the interior of S.

Now let $d \neq 0$ be any integral vector. We will show that

$$w_d(S) \max_{x \in K} d^T x - \min_{x \in K} d^T x \geq n.$$

As $\max_{x \in K} d^T x = -\min_{x \in K} (-d)^T x$, we may assume $d_i \geq 1$ for some i. But then $ne_i, 0 \in S$, where e_i is the i-th unit vector, hence $w_d(K) \geq d^T(ne_i) - d^T 0 \geq n$. Thus S is the desired convex body K.

Exercise 2

Let $P \subseteq \mathbb{R}^n$ be a polyhedron given by the system $Ax \leq b$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
We call a point $p \in P$ a *vertex* of P if there exists an inequality $c^T x \leq \delta$ valid for P such that $\{x \in P : c^T x = \delta\} = \{p\}$.
We call a point $p \in P$ an *extreme point* of P if it cannot be written as the convex combination of other points in P, i.e. there exist no two points $q_1, q_2 \in P$ distinct from p and $\lambda \in (0, 1)$ with $p = \lambda q_1 + (1 - \lambda) q_2$. Show the following.

1. If $x^\star$ is a vertex of P, then $x^\star$ is an extreme point of P.

2. If there is a subsystem $A'x \leq b'$ of $Ax \leq b$ with $A'x^\star = b'$ and $\text{rank}(A') = n$, then $x^\star$ is a vertex of P.

3. If $x^\star$ is an extreme point of P, then there is a subsystem $A'x \leq b'$ of $Ax \leq b$ with $A'x^\star = b'$ and $\text{rank}(A') = n$.

Solution:
Let the notation be as in the exercise.

1. Let $x^\star$ be a vertex of P and let $c^T x \leq \delta$ be the indicating inequality. For the sake of contradiction, assume $x^\star = \lambda y + (1-\lambda)z$ with $y, z \in P$, $\lambda \in (0,1)$. But then

$$c^T x^\star = c^T(\lambda y + (1-\lambda)z) = \lambda c^T y + (1-\lambda)c^T z < \lambda\delta + (1-\lambda)\delta = \delta,$$

   a contradiction.

2. Define $c^T := \mathbf{1}^T A'$ and $\delta := \mathbf{1}^T b'$. We will show that $c^T x \leq \delta$ is feasible for P and is fulfilled with equality only by $x^\star$.

   Obviously, any point $x \in P$ (hence fulfilling $A'x \leq b'$) fulfils $c^T x \leq \delta$ as well, hence the inequality is feasible.

   Now let $p \in P$ be different from $x^\star$. As A' has full rank, there is at least one inequality that is not tight, say the i-th one differs by ε. Then, using nonnegativity of the all-one vector $\mathbf{1}$,

$$c^T p = \mathbf{1}^T(A'p) \leq \mathbf{1}^T(b' - \varepsilon e_i) = \delta - \varepsilon < \delta,$$

   showing that $x^\star$ is the only point fulfilling $c^T x \leq \delta$ with equality.

3. Let $A'x \leq b'$ be a maximum subsystem of $Ax \leq b$ that is fulfilled with equality by $x^\star$, and let $A''x \leq b''$ be the remaining inequalities.

   Let v be a vector in the kernel of A'. Then $x^\star = \left(\frac{1}{2}(x^\star + v) + \frac{1}{2}(x^\star - v)\right)$ expresses $x^\star$ as a convex combination. Since v is in the kernel of A', we have $A'x^\star \pm v \leq b'$ for the first subsystem.

   For the second subsystem, observe that there exists some $\varepsilon > 0$ with

$$A''x^\star \leq b'' - \varepsilon\mathbf{1}.$$

   By scaling v, we may assume $A''(\pm v) \leq \varepsilon\mathbf{1}$ and hence $A''x^\star \pm v \leq b''$.

   As $x^\star$ is an extreme point, this means that v is the zero vector, implying that A' has full rank.

Exercise 3
Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and let $a_1, \ldots, a_n$ denote the columns of A. Show that $\operatorname{cone}(\{a_1, \ldots, a_n\})$ is the polyhedron $P = \{x \in \mathbb{R}^n : A^{-1}x \geq 0\}$.
Show that $\operatorname{cone}(\{a_1, \ldots, a_k\})$ for $k \leq n$ is the set

$$P_k = \{x \in \mathbb{R}^n :\ a_i^{-1}x \geq 0,\ i = 1, \ldots, k;\ a_i^{-1}x = 0,\ i = k+1, \ldots, n\},$$

where a_i^{-1} denotes the i-th row of A^{-1}.

Solution:
Let $k \leq n$, the first statement will follow for $k = n$. First we show $\operatorname{cone}(\{a_1, \ldots, a_k\}) \subseteq P_k$. Let At with $t_{k+1}, \ldots, t_n = 0$ be any conic combination. Then $A^{-1}At = t \geq 0$ with $a_i^{-1}(At) = t_i = 0$.

Now show $P_k \subseteq \operatorname{cone}(\{a_1, \ldots, a_k\})$. Let x s.t. $t := A^{-1}x \geq 0$ with $t_i = a_i^{-1}x = 0$ for all $i > k$. Then $x = AA^{-1}x = At$ is a conic combination of the first k columns.

Exercise 4 [$\star$]
Let $K \subseteq \mathbb{R}^n$ be a convex set. The *polar* of K is defined as

$$K^\star := \{y \in \mathbb{R}^n : \ y^T x \leq 1 \ \forall x \in K\}.$$

(You might want to draw the cube $[-1,1]^2$ *and its polar, and the unit ball and its polar to see what happens. This is not part of the exercise.)*
Show the following.

1. $K^\star$ is a closed convex set that contains the origin.

2. If K is bounded, then $0 \in \operatorname{int} K^\star$.

3. If $0 \in \operatorname{int} K$, then $K^\star$ is bounded.

4. $K_1 \subseteq K_2$ implies $K_1^\star \supseteq K_2^\star$.

5. $K \subseteq (K^\star)^\star$.

6. If K is closed and contains the origin, then $(K^\star)^\star = K$.

Solution:
The polar of the cube is the cross-polytope, the polar of the unit ball is the unit ball itself. Let the notation be as in the exercise.

1. Observe that $K^\star$ is the (infinite) intersection of closed half-spaces. As each half-space is closed and convex, so is $K^\star$. Furthermore, $0^T x = 0 \leq 1$ for all $x \in \mathbb{R}^n$ implies that $K^\star$ contains the origin.

2. Let $K \subseteq B(0,R)$ and consider $y \in B(0,1/R)$. Then for all $x \in K$, one has

$$y^T x \leq ||y||||x|| \leq \frac{1}{R} R = 1.$$

   Hence $B(0,1/R) \subseteq K^\star$.

3. Vice versa, suppose $B(0,\varepsilon) \subseteq K$ and let $y \in K^\star$. Observe that $x := \varepsilon \frac{y}{||y||} \in K$. So we have

$$1 \geq y^T x = \varepsilon ||y|| \quad \Rightarrow \quad ||y|| \leq \frac{1}{\varepsilon},$$

   hence $K^\star$ is bounded.

4. Let $y \in K_2^\star$. Hence y satisfies $y^T x \leq 1$ for all $x \in K_2$, in particular for all $x \in K_1 \subseteq K_2$. Thus $K_2^\star \subseteq K_1^\star$.

5. Pick $x \in K$. By definition of $K^\star$, we have $x^T y \leq 1$ for all $y \in K^\star$. But then $x \in (K^\star)^\star$ by definition of $(K^\star)^\star$.

6. It only remains to show $(K^\star)^\star \subseteq K$ for K closed and containing the origin. For the sake of contradiction, assume there is a point $z \in (K^\star)^\star$ with $z \notin K$. Since K is closed, there is a hyperplane that strictly separates z from K. Formally, there exists an equation $a^T x = b$ such that $a^T x < b$ for all $x \in K$ and $a^T z > b$. Since $0 \in K$, we have $b > 0$. By rescaling, we can ensure $b = 1$, which implies $a \in K^\star$. But since $a^T z > b = 1$, this contradicts $z \in (K^\star)^\star$.