Convexity

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Assignment Sheet 1 - Solutions

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Exercise 1
Let $A, B \subseteq \mathbb{R}^d$ be two convex sets.

1. Show $A + A = 2A$. Does this hold if A is not convex?
(We understand the addition and multiplication of sets to be component-wise, i.e. $A + B = \{a + b \mid a \in A,\ b \in B\}$ and $2A = \{2a \mid a \in A\}$.)

2. Show that $A + B$ is convex.

3. Assume A and B are convex and closed. Show that $\operatorname{conv}(A \cup B)$ is not necessarily closed. Can you give an additional condition to A and B such that $\operatorname{conv}(A \cup B)$ is closed and prove it's sufficiency?

Solution:

1. For the sake of showing $2A \subseteq A + A$, let $2a \in 2A$. Hence, $2a = a + a \in A + A$, showing the first inclusion. Now let $a, b \in A$, hence $a + b \in A + B$. As A is convex, the point $\frac{a+b}{2}$ is contained in A as well. Thus
$$a + b = \frac{a+b}{2} + \frac{a+b}{2} = 2 \cdot \frac{a+b}{2} \in 2A$$
finishes the second inclusion and the proof. If A is not convex, this does not hold, consider for example $A = \{x \in [0,2]^2 | \ x \notin [1,2]^2\}$.

2. Let $a_1, a_2 \in A$, $b_1, b_2 \in B$, $\lambda \in [0,1]$ be arbitrary. Showing $\lambda(a_1 + b_1) + (1 - \lambda)(a_2 + b_2) \in A + B$ finishes the proof. Using convexity of both A and B, we find
$$\lambda(a_1 + b_1) + (1 - \lambda)(a_2 + b_2) = \underbrace{\lambda a_1 + (1 - \lambda)a_2}_{\in A} + \underbrace{\lambda b_1 + (1 - \lambda)b_2}_{\in B} \in A + B.$$

3. A possible counterexample is $A = \{0\}$ and $B = \{x \in \mathbb{R}^n | \ x^T e_1 \geq 1\}$ for the first canonic unit vector e_1. We claim $\operatorname{conv}(A \cup B) = C$ with $C := \{0\} \cup \{x \in \mathbb{R}^n | \ x^T e_1 > 0\}$: any $p \in \operatorname{conv}(A \cup B)$ is of the form λb with $b \in B$, $\lambda \in [0,1]$. As $b^T e_1 \geq 1$, we have $\lambda b^T e_1 > 0$ for any $\lambda \neq 0$. But for $\lambda = 0$, we have $\lambda b = 0$, implying $\operatorname{conv}(A \cup B) \subseteq C$. For the other implication, pick $p \in C$. If $p = 0 \in A$, or $1 \leq p^T b$ (implying $b \in B$), we are done. Otherwise, set $\delta = p^T e_1 \in (0,1)$. But then $p = (1 - \delta)0 + \delta(\frac{1}{\delta}p) \in \operatorname{conv}(A \cup B)$, proving the claim. Obviously, C is not closed.

   If A and B are bounded on the other hand, $\operatorname{conv}(A \cup B)$ is closed, as we will see now. Take any sequence $\{q_i\}_{i \in \mathbb{N}} = \{\lambda_i a_i + (1 - \lambda_i) b_i\}_{i \in \mathbb{N}}$ in $\operatorname{conv}(A \cup B)$ that converges and call it's limit p. We will show $p \in \operatorname{conv}(A \cup B)$, implying that $\operatorname{conv}(A \cup B)$ is closed.

   Recall the lemma of Bolzano-Weierstrass, stating that each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. As A is bounded, the sequence $\{a_i\}_{i \in N}$ has a convergent subsequence, say $\{a_j\}_{j \in J}$ (with $J \subseteq \mathbb{N}$). Using the fact that B is bounded, we can find a convergent subsequence of

$\{b_j\}_{j\in J}$, call it $\{b_i\}_{i\in I}$ (with $I \subseteq J$). Of course, $\{a_i\}_{i\in I}$ converges as well. As a last step we look for a convergent subsequence in $\{\lambda_i\}_{i\in I}$, call the index set $K \subseteq \mathbb{N}$.

As A, B and $[0,1]$ are closed, we can find limits $a^\star \in A$, $b^\star \in B$ and $\lambda^\star \in [0,1]$ of our subsequences. Now the initial sequence converges to $\lambda^\star a^\star + (1-\lambda^\star)b^\star$, which is contained in the convex hull.

Hence, $\mathrm{conv}(A \cup B)$ is closed.

Exercise 2 [$\star$]

1. Let $K \subseteq \mathbb{R}^n$ be centrally symmetric, convex, compact and of positive volume. For any such K, define $\|\cdot\|_K : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ as
$$\|x\|_K := \min\{r \geq 0 \mid x \in rK\}.$$
Show that $\|\cdot\|_K$ is a norm.

2. Let $\|\cdot\|$ be any norm. Show that this norm is induced by a centrally symmetric, convex, compact set $K \subset \mathbb{R}^n$ of positive volume, i.e. there exists some K s.t.
$$\|\cdot\| = \|\cdot\|_K.$$

Solution:

1. Let K be as in the exercise. We have to show

   - $\|x\|_K$ is well defined for all $x \in \mathbb{R}^n$,
   - $\|x\|_K = 0 \Rightarrow x = 0$,
   - $\|\lambda x\|_K = |\lambda| x$ for any scalar λ and all $x \in \mathbb{R}^n$, and
   - the triangle inequality holds, i.e. $\|x+y\|_K \leq \|x\|_K + \|y\|_K$ for all $x, y \in \mathbb{R}^n$.

   First of all, let us show that the norm is well defined. As K has positive volume, there exists some $\epsilon > 0$ s.t. $B(0,\epsilon) \subseteq K$. For any $x \in \mathbb{R}^n$, this implies $x \in \frac{\|x\|_2}{\epsilon}K$. This shows that K is well defined.

   As K is bounded, for each $0 \neq x \in \mathbb{R}^n$ there exists $\lambda > 0$ such that $\lambda x \notin K$, implying $\|x\|_K \geq \frac{1}{\lambda}$. This proves the second property.

   Let us show linearity for scalar multiplication. By definition and using symmetry, we find
$$\begin{aligned}\|\lambda x\|_K &= \min\{r \geq 0 \mid \lambda x \in rK\}\\ &= \min\{r \geq 0 \mid x \in \frac{r}{|\lambda|}K\}\\ &= \min\{|\lambda| r \geq 0 \mid x \in rK\}\\ &= |\lambda|\|x\|_K.\end{aligned}$$

   It remains to show the triangle inequality. Let $x, y \in \mathbb{R}^n$ and note that $\frac{x}{\|x\|_K} \in K$ and $\frac{y}{\|y\|_K} \in K$. We are done once we have shown $x + y \in (\|x\|_K + \|y\|_K)K$.
$$x + y = (\|x\|_K + \|y\|_K)\underbrace{\left(\frac{\|x\|_K}{\|x\|_K + \|y\|_K}\frac{x}{\|x\|_K} + \frac{\|y\|_K}{\|x\|_K + \|y\|_K}\frac{y}{\|y\|_K}\right)}_{\in K \text{ by convexity of } K} \in (\|x\|_K + \|y\|_K)K.$$

2. We claim that the set K we are looking for is the unit ball w.r.t. the given norm,

$$K = B_{\|\cdot\|}(0,1) = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}.$$

Let us first check the properties for K.

It is clear that K is closed. As $\|x\| = \|-x\|$, it is centrally symmetric. Since any vector has strictly positive norm, K is bounded (since there exists some λ s.t. $\|\lambda x\| = |\lambda|\|x\| > 1$).

Moreover, for any basis $b_1, \dots, b_n$ of $\mathbb{R}^n$ we can find $\epsilon > 0$ s.t. $\epsilon\|b_i\| \le 1$, hence K is of positive volume.

For $x, y \in K$, $\lambda \in [0,1]$, convexity follows by the triangle inequality of the norm:

$$\|\lambda x + (1-\lambda)y\| \le \lambda\|x\| + (1-\lambda)\|y\| \le \lambda \cdot 1 + (1-\lambda)\cdot 1 = 1.$$

It remains to show that the norm induced by K is the same as the given norm. Let $0 \ne x \in \mathbb{R}^n$ with $\|x\| = s$.

Then

$$\begin{aligned} \|x\|_K &= \min\{r \ge 0 \mid x \in rK\} \\ &= \min\{r \ge 0 \mid \|x\| \le r\} \\ &= \min\{r \ge 0 \mid s \le r\} \\ &= s, \end{aligned}$$

finishing the proof.

Exercise 3

Prove the following variant of *Minkowski's theorem* (You may use Minkowski's theorem as seen in class):

Let $C \subseteq \mathbb{R}^d$ be symmetric around the origin, convex, closed and bounded, and suppose that $\mathrm{vol}(C) \ge 2^d$. Then C contains at least one integer point different from 0.

Solution:

Solution 1: Suppose for contradiction the claim does not hold. Then there $\exists K \subset \mathbb{R}^d$ that is compact and centrally symmetric, with $vol(K) = 2^d$ that contains no lattice point other than the origin.

Say K is contained in a ball of radius B. Now look at the set $S = \{x \in \mathbb{Z}^d \backslash \{0\} : \|x\| < 2B\}$ and consider the function $f : S \to \mathbb{R}$: $x \mapsto d(x, K)$, the function that assigns to each point in S its distance from the set K. Since K is compact, there exists $x \in S$ and $y \in K$ s.t. $\|x^\star - y^\star\| > 0$ and $\|x - y\| \ge \|x^\star - y^\star\|$ for all $x \in S, y \in K$. In fact, by the choice of S, $\|x - y\| \ge \|x^\star - y^\star\|$ for all $x \in \mathbb{Z}^d \backslash \{0\}, y \in K$.

So the minimal distance between K and any non-zero lattice point is positive, say greater than $\epsilon > 0$. Now, we can "blow up " K by a small amount so that its volume will be strictly greater then 2^d, but it still wouldn't contain any non-zero lattice point. More specifically, let $K' = (1 + \frac{\epsilon}{2B})K$. Then

- K' is compact, convex and centrally symmetric
- $vol(K') = \left(1 + \frac{\epsilon}{2B}\right)^d vol(K) > 2^d$

- For all $(1+\frac{\epsilon}{2B})y \in K', x \in \mathbb{Z}^d\backslash\{0\}$

$$\|x-(1+\frac{\epsilon}{2B})y\| \quad \geq \quad \|x-y\|-\|y-(1+\frac{\epsilon}{2B})y\| \geq \epsilon - \frac{\epsilon}{2B}\|y\| \geq \frac{\epsilon}{2} > 0$$

so in particular, $K' \cap \mathbb{Z}^d\{0\} = \emptyset$

This contradicts Minkowski's theorem.

Solution 2: By Minkowski theorem, we know that the set $(1+\frac{1}{n})K$ contains a non-zero lattice point x_n for all $n \in \mathbb{N}$. Now, since K is compact and $x_n \in 2K, \forall n \in N$, the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_j} \to x$, say.

Since $\Lambda \subset \mathbb{R}^d$ is also a closed set, $x \in \Lambda$. Moreover,

$$d(x_n, K) \leq d\left(x_n, \frac{n}{n+1}x_n\right) = \frac{1}{n+1}\|x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

where the first inequality holds since $\frac{n}{n+1}x_n \in K$, and the convergence follows from the fact that $2K$ is bounded. But then $d(x, K) = 0$, and K is closed, so in fact $x \in K$.

The condition "K closed" is necessary: The following example shows that, in fact, we need K to be closed. Let $K = \{x \in \mathbb{R}^d : \|x\|_\infty < 1\} \subset \mathbb{R}^d$. K is then convex, bounded and centrally symmetric, and $vol(K) = 2^d$, but it contains no lattice point other than the origin.

Exercise 4

Give a proof of *Caratheodory's theorem*:

Let $\mathrm{X} \subseteq \mathbb{R}^d$. Then each point in $conv(\mathrm{X})$ is in $conv(\mathrm{S})$ for some $\mathrm{S} \subseteq \mathrm{X}$, $|\mathrm{S}| \leq d+1$.

Solution:

Let $x \in conv(\mathrm{X})$. Pick $k \in \mathbb{N}$ minimal, such that x can be written as a convex combination of k elements in X. That is, pick k such that

- we cannot write x as a convex combination of any $k-1$ elements in X.
- $x = \sum_{i=1}^k \lambda_i x_i$ for some $x_1, \dots, x_k \in \mathrm{X}$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ with $\sum_{i=1}^k \lambda_i = 1$.

Suppose for contradiction that $k > d+1$.

Since $k \geq d+2$, by Radon's theorem there exist $I \subset \{1, \dots, k\}$ and $y \in \mathbb{R}^d$ s.t. $y \in conv(\{x_i : i \in I\}) \cap conv(\{x_i : i \in [k]\backslash I\})$. In particular, there exist $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ s.t. $\sum_{i=1}^k \alpha_i = 0$ and $\sum_{i=1}^k \alpha_i x_i = 0$. Then, if we set $\epsilon = min\{\frac{\lambda_i}{|\alpha_i|} : \alpha_i < 0\}$, we can express x as

$$x = \sum_{i=1}^k \lambda_i x_i + \epsilon \sum_{i=1}^k \alpha_i x_i = \sum_{i=1}^k (\lambda_i + \epsilon\alpha_i) x_i$$

as a convex combination of strictly less than k elements in X, a contradiction.