Convexity

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Assignment Sheet 9 - Solutions

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Exercise 1

The Gamma function if defined as $\Gamma(x) = \int_0^\infty r^{x-1}e^{-r}\mathrm{d}r$ for $x > 0$. Prove that

1. $\Gamma(x+1) = x\Gamma(x)$.
2. $\Gamma(n) = (n-1)!$ for positive integer n.

Solution:

1. We integrate $\Gamma(x+1)$ by parts to get

$$\Gamma(x+1) = \int_0^\infty r^x e^{-r}\mathrm{d}r = [-r^x e^{-r}]_0^\infty + \int_0^\infty (xr^{x-1}e^{-r})\mathrm{d}r = x\Gamma(x)$$

2. Applying (1.) repeatedly gives $\Gamma(n) = (n-1)!\Gamma(1) = (n-1)!$, since $\Gamma(1) = \int_0^\infty e^{-r}\mathrm{d}r = 1$

Exercise 2

Prove the missing step in the computation of v_n, the volume of the ball B_1^n, from the lecture. Namely show that

$$\int_0^\infty nR^{n-1}e^{-\frac{R^2}{2}}\mathrm{d}R = 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)$$

Solution:

Apply change of variables $r = \frac{R^2}{2}$. Then $\mathrm{d}r = R\mathrm{d}R$ and

$$\begin{aligned}\int_0^\infty nR^{n-1}e^{-\frac{R^2}{2}}\mathrm{d}R &= \int_0^\infty n2^{\frac{n}{2}-1}\left(\frac{R^2}{2}\right)^{\frac{n}{2}-1}e^{-\frac{R^2}{2}}(R\ \mathrm{d}R) = n2^{\frac{n}{2}-1}\int_0^\infty r^{\frac{n}{2}-1}e^{-r}\mathrm{d}r = \\ &= \frac{n}{2}2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) = 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)\end{aligned}$$

Exercise 3 [$\star$]

Let $A, B \subseteq \mathbb{R}^n$ be convex sets. Show the following equality of sets (in $\mathbb{R}^{n+1}$).

$$\mathrm{conv}\left((\{0\}\times A)\cup(\{1\}\times B)\right) \quad = \quad \bigcup_{t\in[0,1]}\left[\{t\}\times((1-t)A+tB)\right].$$

Solution:

Denote the set on the left-hand side by C_1, the one on the right-hand side by C_2.

- $C_1 \subseteq C_2$: Let $(t,x) = \sum_{i\in I} \alpha_i(0,a_i) + \sum_{j\in J} \beta_j(1,b_j) \in C_1$ be a convex combination with $\forall i \in I: \; a_i \in A$ and $\forall j \in J: \; b_j \in B$. Considering the first coordinate, we have $\sum_{j\in J} \beta_j = t$ and hence $\sum_{i\in I} \alpha_i = (1-t)$. Moreover, as all coeffitients are non-negative, we have $\forall j \in J: \; \beta_j \leq t$ and $\forall i \in I: \; \alpha_i \leq (1-t)$. But then

$$\hat{a} = \sum_{i\in I} \frac{\alpha_i}{1-t} a_i \in A \qquad \text{and} \qquad \hat{b} = \sum_{j\in J} \frac{\beta_j}{t} b_j \in B$$

  are convex combinations and $(t,x) = (t,(1-t)\hat{a} + t\hat{b} \in \{t\} \times ((1-t)A + tB)$, implying $C_1 \subseteq C_2$.

- $C_2 \subseteq C_1$: There is nothing to show, as for $(t,x) \in C_2$, we have $(t,x) = (t,(1-t)a + tb) = (1-t)(0,a) + t(1,b) \in C_1$.

Exercise 4
Let $A \subseteq \mathbb{R}^n$ be a brick set consisting of at least two bricks. Show that there exist a canonic unit vector $e_i \in \mathbb{R}^n$ and $b \in \mathbb{R}$ and two bricks $B_1, B_2 \in A$ s.t. $e_i^\top x \leq b$ for all $x \in B_1$ and $e_i^\top x \geq b$ for all $x \in B_2$. That is, show there exists a hyperplane that separates two bricks of A completely.

Solution:
Let $B_1 = \{x : a_i \leq x_i \leq b_i, \; \forall i = 1,\ldots,d\}$ and $B_2 = \{x : c_i \leq x_i \leq d_i, \; \forall i = 1,\ldots,d\}$ be two bricks in A for some reals with $a_i < b_i, c_j < d_j$ for all i, j. If there is some i such that either $b_i \leq c_i$, then the two bricks can be separated by the hyperplane $\{x \in \mathbb{R}^n : x_i = b_i\}$ and similarly if $d_i \leq a_i$. So suppose this is not the case. Then, for each i, there must be a $y_i \in \mathbb{R}$ with $y_i \in (a_i,b_i) \cap (c_i,d_i)$. But then the point $y = (y_1,\ldots,y_n)$ lies in the interiors of both B_1 and B_2, contradicting the assumption that the bricks have pairwise disjoint interiors.