Convexity

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Assignment Sheet 11 - Solutions

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Exercise 1
For $\lambda > 0$, show $\sum_{i=1}^{\infty} \frac{\lambda^{2i}}{(2i)!} \leq e^{\lambda^2/2}$ which we used to show the Chernoff bound.

Solution:
We know that the exponential function coincides with its Taylor series around 0, i.e.

$$e^x = \sum_{i=0}^{\infty} \frac{(x-0)^i}{i!} e^0 = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

(assuming $0! = 1$). Hence, we have

$$e^{\lambda^2/2} = \sum_{i=1}^{\infty} \frac{\lambda^{2i}}{2^i i!} \geq \sum_{i=1}^{\infty} \frac{\lambda^{2i}}{(2i)!}.$$

Exercise 2
Let $N(\mu, \sigma^2)$ denote the density function $f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(X-\mu)^2}{\sigma^2}}$. Prove the following Lemma.

Let $X_1, \ldots, X_n$ be independent random variables with density function $N(0,1)$ and let $a = (a_1, \ldots, a_n)^\intercal \in \mathbb{R}^n$. Then $\sum_{i=1}^{n} a_i X_i$ has density function $N(0, \|a\|_2^2)$.

[*Hint: Start with only two independent random variables X and Y. How does the density function for (X,Y) look like? What happens if you apply a rotation to (X,Y)? Can you use this to show $c_1X + c_2Y \sim N(0,1)$ for constants c_1, c_2 with $c_1^2 + c_2^2 = 1$? For $a > 0$, how does the density function for aX look like? Can you go on from there?*]

Solution:
First, consider only two variables x, y. As x, y are independent, the probability distribution of (x,y) is just the product of their single distributions,

$$\frac{1}{\sqrt{2\pi}} e^{-1/2x^2} \frac{1}{\sqrt{2\pi}} e^{-1/2y^2} = \frac{1}{2\pi} e^{-1/2\|(x,y)\|^2} = 1.$$

If we rotate (x,y) by ϕ around 0, its density function is

$$f((x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi)) = \frac{1}{2\pi} e^{-1/2\|(x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi)\|^2}.$$

An easy calculation shows $\|(x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi)\|^2 = \|(x,y)\|^2$ (using $\sin^2 + \cos^2 = 1$), hence the density function is invariant under rotation. This means, considering only the second coordinate of $(x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi)$, that $c_1x + c_2y \sim N(0,1)$ whenever $c_1^2 + c_2^2 = 1$.

Furthermore, substituting $z = cx$ and computing its density function ($\frac{1}{\sqrt{2\pi}|c|} e^{-1/2(z/c)^2}$) shows that $cx \sim N(0, c^2)$.

These two things together mean that if c_1, c_2 are s.t. $c_1^2 + c_2^2 = c^2$ for some c, then $\frac{c_1}{c}X + \frac{c_2}{c}Y \sim N(0,1)$, and so $c_1 X + c_2 Y \sim N(0, c^2)$. Combining the above facts, we will derive the distribution of $\sum_{i=1}^n a_i X_i$ by induction on n.

We know that the claim holds for $n = 2$. So suppose now $n \geq 3$ and that the claim holds for $n-1$. Let us denote $s_k^2 = \sum_{i=1}^k a_i^2$ for $k = 1, \ldots, n$. Then we can write

$$\sum_{i=1}^n a_i X_i = s_n \left(\sum_{i=1}^{n-1} \left(\frac{a_i}{s_n} X_i \right) + \frac{a_n}{s_n} X_n \right) = s_n \left(\frac{s_{n-1}}{s_n} \sum_{i=1}^{n-1} \left(\frac{a_i}{s_{n-1}} X_i \right) + \frac{a_n}{s_n} X_n \right)$$

If we now denote $Y_{n-1} = \sum_{i=1}^{n-1} \frac{a_i}{s_{n-1}} X_i$, we note that it is a sum of $n-1$ scaled independent standard gaussians, with $\sum_{i=1}^{n-1} \left(\frac{a_1}{s_{n-1}} \right)^2 = 1$, so by induction hypothesis, $Y_{n-1} \sim N(0,1)$. Next, $\frac{s_{n-1}}{s_n} Y_{n-1} + \frac{a_n}{s_n} X_n \sim N(0,1)$ for the same reason. Finally, $s_n \left(\frac{s_{n-1}}{s_n} Y_{n-1} + \frac{a_n}{s_n} X_n \right) \sim N(0, s_n^2)$, as required.

Exercise 3
Let $X \sim N(0,1)$. Find a constant $a > 0$ such that $\Pr(X > \lambda) \leq e^{-a\lambda^2}$ for all $\lambda > 0$.

Solution:
First, for $\lambda \geq \frac{1}{\sqrt{2\pi}}$ we have that

$$\begin{aligned} \Pr(X \geq \lambda) &= \int_\lambda^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \leq \int_\lambda^\infty \frac{x}{\lambda} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= \left[-\frac{1}{\sqrt{2\pi}\lambda} e^{-\frac{x^2}{2}} \right]_\lambda^\infty = \frac{1}{\sqrt{2\pi}\lambda} e^{-\frac{\lambda^2}{2}} \leq e^{-\frac{\lambda^2}{2}} \end{aligned}$$

So let's prove the same bound works for $0 < \lambda < \frac{1}{\sqrt{2\pi}}$ as well. We want to show that $f(\lambda)$ is negative on $(0, \frac{1}{\sqrt{2\pi}})$ where $f(\lambda) = \Pr(X \geq \lambda) - e^{\frac{-\lambda^2}{2}}$. First note that $f(0) < 0$, so it suffices to show that f is decreasing on $(0, \frac{1}{\sqrt{2\pi}})$. But this is clear, since $f(\lambda) = 1 - \int_{-\infty}^\lambda \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx - e^{-\frac{1}{2}\lambda^2}$, and $f'(\lambda) = \lambda e^{-\frac{\lambda^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} \leq 0$.

Alternatively for $\lambda > 0$,

$$\Pr(X \geq \lambda) = \Pr(e^{\lambda X} \geq e^{\lambda^2}) \leq \frac{\mathbb{E}(e^{\lambda X})}{e^{\lambda^2}}$$

where the last inequality follows by Markov inequality. It remains to show that $\mathbb{E}(e^{\lambda X}) = e^{\frac{\lambda^2}{2}}$.

Exercise 4
Show the other estimation for the Gaussian Annulus theorem. This is, show the following.
For $X \sim N(0,1), X \in \mathbb{R}^n$, show that $\Pr[\|X\| \leq \sqrt{n} - \beta] \leq e^{-\frac{c\beta^2}{2}}$, where $\beta > 0$ and c is a global constant.

Solution:
Assume $\beta \leq \sqrt{n}$ and set $Q = \sqrt{n} - \beta$. First note

$$\begin{aligned} \mathbb{E}[e^{-\lambda\|X\|^2}] &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\lambda\|X\|^2} e^{-1/2\|X\|^2} dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-(\lambda+1/2)\|X\|^2} dx \qquad \text{substituting } z = \sqrt{2\lambda+1}X \\ &= (2\lambda+1)^{-n/2}. \end{aligned}$$

Now calculate

$$
\begin{aligned}
\Pr[\|X\| \le \sqrt{n} - \beta] &= \Pr[\|X\|^2 \le Q^2] \\
&= \Pr[-\|X\|^2 \ge -Q^2] \\
&= \Pr[e^{-\lambda\|X\|^2} \ge e^{-\lambda Q^2}] \\
&= \Pr\left[e^{-\lambda\|X\|^2} \ge \frac{e^{-\lambda Q^2}\mathbb{E}[e^{-\lambda\|X\|^2}]}{\mathbb{E}[e^{-\lambda\|X\|^2}]}\right] \\
&\le \frac{\mathbb{E}[e^{-\lambda\|X\|^2}]}{e^{-\lambda Q^2}} \qquad\qquad \text{by Markov} \\
&= \frac{(2\lambda+1)^{-n/2}}{e^{-\lambda Q^2}} \\
&\le e^{-\lambda n - \lambda Q^2} \\
&= e^{-\beta^2/2} \qquad\qquad \text{by setting } \lambda = \frac{\beta^2}{2(n+Q^2)} = \frac{\beta^2}{4n + 2\beta^2 - 4\sqrt{n}\beta}.
\end{aligned}
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