Convexity

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Assignment Sheet 10 - Solutions

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Exercise 1 [$\star$]
Consider the following two statements.

(i) [The Brunn-Minkowski inequality.] For non-empty, compact $A, B \subseteq \mathbb{R}^n$,

$$\text{vol}(A+B)^{1/n} \geq \text{vol}(A)^{1/n} + \text{vol}(B)^{1/n}$$

(ii) For all compact $C, D \subseteq \mathbb{R}^n$ and all $t \in (0,1)$,

$$\text{vol}((1-t)C + tD) \geq \text{vol}(C)^{1-t}\,\text{vol}(D)^t.$$

1. Derive (ii) from (i); prove and use the inequality $(1-t)x + ty \geq x^{1-t}y^t$ positive reals $x, y \in \mathbb{R}$ and $t \in (0,1)$.
   [*Hint: For $t \in \{0,1\}$, the inequality is obviously true. Is there another value for t for which the inequality is easy to show? Can you go on from there?*]

2. Prove (i) from (ii).
   [*Hint: What is the problem with showing the reverse direction? What happens if two sets C', D' have the same volume? What can you do with C and D to get this property?*]

Solution:

1. If any of C, D has measure 0, (ii) is trivially true, hence let us assume they both have positive measure.

   If the stated inequality is true, Brunn-Minkowski gives us immediately

$$\begin{aligned} \text{vol}\left((1-t)A + tB\right)^{1/n} &\geq \text{vol}\left((1-t)A\right)^{1/n} + \text{vol}\left(tB\right)^{1/n} \\ &= (1-t)\,\text{vol}(A)^{1/n} + t\,\text{vol}(B)^{1/n} \\ &\geq \text{vol}(A)^{(1-t)/n}\,\text{vol}(B)^{t/n}. \end{aligned}$$

   It is left to show the inequality.

   One way is using Young's inequality, stating that for $p, q > 1$ with $1/p + 1/q = 1$ and $a, b \geq 0$ one has

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}.$$

   Setting $p = 1/(1-t)$, $q = 1/t$, $a = x^{1/p}$, $b = y^{1/q}$, all requirements are met and we get immediately the desired inequality.

   Another proof uses nested intervals. Obviously, it holds for $t = 0, 1$. For $t = 1/2$, we find $(1/2x - 1/2y)^2 \geq 0$ implying $1/2x^2 + 1/2y^2 \geq xy$. Substituting $\bar{x} = x^2$ and $\bar{y} = y^2$ finishes the calculation. By induction the claim is true for any $\alpha 2^{-k}$ with $1 \leq \alpha \leq 2^k - 1$, $\alpha \in \mathbb{Z}$, as the following calculation shows:

$$\begin{aligned}
(1-\alpha 2^{-k-1})x + \alpha 2^{-k-1} y &= \frac{1}{2}\left(\left((2-\alpha 2^{-k})x + \alpha 2^{-k} y\right)\right) \\
&\geq \left((1-\left\lfloor\frac{\alpha}{2}\right\rfloor 2^{-k})x + \left\lfloor\frac{\alpha}{2}\right\rfloor 2^{-k} y\right)^{1/2} \left((1-\left\lceil\frac{\alpha}{2}\right\rceil 2^{-k})x + \left\lceil\frac{\alpha}{2}\right\rceil 2^{-k} y\right)^{1/2} \\
&\geq x^{1-\alpha 2^{-k-1}} y^{\alpha 2^{-k-1}}.
\end{aligned}$$

Defining $t_k \in 2^{-k}\mathbb{Z}$ s.t. $t_k \leq t < t_k + 2^{-k}$ for $k \in \mathbb{N}$ gives a sequence converging to t. As the inequality holds for every member of the sequence, it also holds for the limit.

2. First note that if A or B has measure 0, the Brunn-Minkowski inequality is trivially true, hence assume again positive measure. If both sets have positive volume, (ii) holds for $t = 0, 1$ as well. We first claim that there exists some $t^\star \in [0, 1]$ s.t. $\operatorname{vol}((1-t)A) = \operatorname{vol}(tB)$. This holds as the function $v(t) = \operatorname{vol}((1-t)A) - \operatorname{vol}(tB)$ is a polynomial in t, hence continuous, and fulfils $v(0) > 0$ and $v(1) < 0$. Let $A^\star = (1-t^\star)A,\ B^\star = t^\star B$, thus

$$\begin{aligned}
A &= \frac{1}{1-t} A^\star \\
B &= \frac{1}{t} B^\star \\
v^\star &:= \operatorname{vol} A^\star = \operatorname{vol} B^\star
\end{aligned}$$

Now

$$\begin{aligned}
\operatorname{vol}(A)^{1/n} + \operatorname{vol}(B)^{1/n} &= \left(\frac{1}{1-t}\right) v^{\star 1/n} + \left(\frac{1}{t}\right) v^{\star 1/n} \\
&= \frac{1}{t(1-t)} v^{\star 1/n} \\
&= \left[\left(\frac{1}{t(1-t)}\right)^n \operatorname{vol}(A^\star)^t \operatorname{vol}(B^\star)^{1-t}\right]^{1/n} \\
&\leq \left[\left(\frac{1}{t(1-t)}\right)^n \operatorname{vol}\left(tA^\star + (1-t)B^\star\right)\right]^{1/n} \\
&= \operatorname{vol}\left(\frac{1}{1-t} A^\star + \frac{1}{t} B^\star\right)^{1/n} \\
&= \operatorname{vol}\left(A + B\right)^{1/n}
\end{aligned}$$

Exercise 2
Let $A, B \subseteq S^{n-1}$ be measurable sets with distance at least $2t$. Define a measure $\mu(C) = \operatorname{vol}(C) s_{n-1}^{-1}$, where $s_{n-1} = \operatorname{vol}_{n-1}(S^{n-1})$ is the volume of the sphere for any measurable set $C \subseteq S^{n-1}$.
Prove that $\min\{\mu(A), \mu(B)\} \leq 2e^{-t^2 n/4}$.

Solution:
As A, B have distance at least $2t$ the set $A_t \cap B_t$ is a set of measure 0. Hence, $\mu(A_t) + \mu(B_t) \leq 1$. Let w.l.o.g. $\mu(A_t) \leq 1/2$. Then for the complement of A_t we have $\mu(A_t^c) \geq 1/2$ and by the results of the lecture it follows that

$$\min\{\mu(A), \mu(B)\} \leq \mu(A) \leq \mu(A_t) = 1 - \Pr(A_t^c) \leq 2e^{-t^2 n/4}.$$

Exercise 3
Let E be an equator of the unit ball B_1^n and let A_t be a belt of width $2t$ around E for $t \in (0,1)$. Formally, $E = \{x \in S^{n-1} : a^\intercal x = 0\}$ and $A_t = \{x \in S^{n-1} : |a^\intercal x| \leq t\}$ for some $a \in \mathbb{R}^n \setminus \{0\}$.

Show that if $\Pr(A_t) = \frac{1}{2}$, then $t = O(n^{-\frac{1}{2}})$, that is, half of the measure on the sphere is concentrated in the strip of width $O(n^{-\frac{1}{2}})$ around an equator.

[*Hint:* Recall the measure concentration inequality on a sphere, that we saw on the lecture, for subset $X \in S^{n-1}$ with $\Pr(X) \geq \frac{1}{2}$. Apply this to the two halfspheres defined by the equator.]

Solution:
First, note that if we want to write that the belt of width t has the form as above, we want to say the distance of the points in A_t from the equator hyperplane is at most t. For that we either assume a is the *unit* normal of the equator, in which case it holds that $A_t = \{x \in S^{n-1} : |a^\intercal x| \leq t\}$, or we write $A_t = \{x \in S^{n-1} : \frac{|a^\intercal x|}{\|a\|} \leq t\}$.

So for simplicity assume a is a normal vector of the equator hyperplane H.

Let X and Y be the upper and lower halfspheres, respectively, such that $X \cap Y = E$ and $X \cup Y = S^{n-1}$. Then $\Pr(X) = \Pr(Y) = \frac{1}{2}$. Let $\tilde{t}$ be such that $\Pr(X_{\tilde{t}}), \Pr(Y_{\tilde{t}}) \geq \frac{3}{4}$. Using the bound from the lectures, we obtain that it is sufficient to take $\tilde{t}$ that satisfies

$$\frac{1}{4} \leq 1 - \Pr(X_{\tilde{t}}) \leq 2e^{-\frac{n\tilde{t}^2}{4}} \qquad \text{i.e.} \qquad \tilde{t} = \sqrt{\frac{4\log 8}{n}}$$

It remains to come up with t such that $X_{\tilde{t}} = X \cup A_t$ and $Y_{\tilde{t}} = Y \cup A_t$. Having done that, we know that $\Pr(A_t) = \Pr(X_{\tilde{t}}) + \Pr(Y_{\tilde{t}}) - 1 \geq \frac{1}{2}$.

Using basic geometry we see that if t be chosen to satisfy

a) $\tilde{t}^2 = t^2 + x^2$

b) $1 = t^2 + (1-x)^2 = t^2 + x^2 + 1 - 2x$

then $A_t \subset X_{\tilde{t}}$, and $\Pr(A_t) \geq \frac{1}{2}$. Having done that, we will be able to infer that (at least) half of the measure of the sphere lies in this strip of width t.

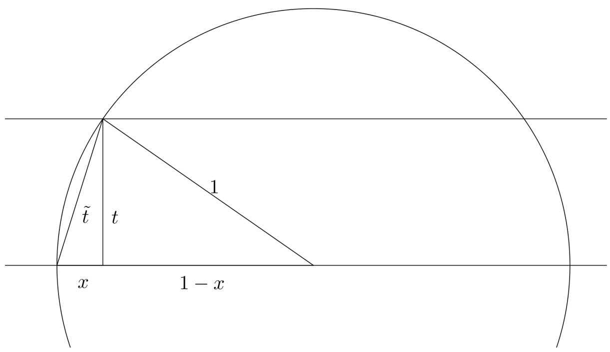

Figure 1: Relationship between t and $\tilde{t}$

From the two equations we get that $x = \frac{1}{2}\tilde{t}^2 = \frac{1}{n}2\log 8$ and

$$t^2 = \tilde{t}^2 - x^2 = \frac{1}{n}4\log 8 - \frac{1}{n^2}4\log^2 8$$

And hence $t = O(\frac{1}{\sqrt{n}})$ as required.