Optimization Methods in Finance
Part 1
Welcome
Welcome!

- My name is Friedrich Eisenbrand
- Assistants of the course: Thomas Rothvoß, Nicolai Hähnle

How to contact me:

- Come to see me during office ours: Tuesdays 11-12
- Send me an e-mail friedrich.eisenbrand@epfl.ch at least one day ahead, announcing your visit
- Apart from personal contact: I cannot guarantee to answer all your e-mails.
- **Highly appreciated:** E-mails about errors/typo’s on my slides

Course webpage

See http://disopt.epfl.ch and follow the Teaching link!
Syllabus

Topics

- Linear Programming: Simplex Method, Computing a dedicated bond portfolio, asset pricing
- Quadratic Programming: Portfolio Optimization (Markowitz model)
- Integer Programming: Constructing an index fund, Combinatorial Auctions
- Dynamic Programming: Option Pricing, Structuring asset backed securities
- Stochastic Programming: Asset/Liability management
- Convex Programming: Market Equilibria Problems
Prerequisites and literature

Prerequisites

- Linear Algebra
- Basic programming skills: C++, Matlab
- Basic knowledge in Algorithms: $O$-Notation, Pseudocode, …
- Basic probability theory

Literature

- D. Luenberger, *Investment Science*, Oxford University Press (Reference on basic mathematics and terminology in finance)
Main skills

- Learning of basic optimization methods and how to apply them in the world of finance: Modeling, practical problem solving (case studies) using software packages
- Mathematical rigor: Proofs, analysis of running time, correctness of methods
## Organization

### Exercises

- Every two weeks there is an assignment sheet.
- For the $i$-th exercise sheet the students are supposed to solve the assignments in the tutorial in week $2i + 1$. The solutions will be discussed in week $2i + 2$.
- First assignment is available in the 2nd week.
## Organization

<table>
<thead>
<tr>
<th>Grading</th>
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<tbody>
<tr>
<td>▶ A total amount of 100 points can be reached as follows</td>
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<tr>
<td>▶ 30 points from the midterm</td>
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<tr>
<td>▶ 10 points from the presentation of a theoretical exercise</td>
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<tr>
<td>▶ 20 points for practical exercises</td>
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<tr>
<td>▶ 40 points from the final exam</td>
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<td>▶ The 10 points can already be gained by presenting a single theoretical exercise</td>
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<td>▶ 60 points suffice to pass the course</td>
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<td>▶ This is a 4-credit course</td>
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Part 2

Linear Programming
Primary objectives:

- Linear programming: Example, Definition
- The standard form
- Basic feasible solutions
- Simplex algorithm
- Along the way: Develop your own toy-solver in C++
- Use of modeling languages and efficient LP-solvers
- Constructing dedicated bond-portfolio
Example: Modelling Cashflow Management

Our company has net cash flow requirements (in 1000 CHF)

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>April</th>
<th>May</th>
<th>June</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net cash flow</td>
<td>-150</td>
<td>-100</td>
<td>200</td>
<td>-200</td>
<td>50</td>
<td>300</td>
</tr>
</tbody>
</table>

For example in January we have to pay CHF 150k, while in March we will get CHF 300k. Initially we have 0 CHF, but the following possibilities to borrow/invest money

1. We have a credit line of CHF 100k per month. For a credit we have to pay 1% interest per month.

2. In the first 3 months we may issue a 90-days commercial paper (to borrow money) at a total interest of 2% (for the whole 3 months).

3. Excess money can be invested at an interest rate of 0.3% per month.

Task: We want to maximize our wealth in June, while we have to fulfill all payments. How shall we invest/borrow money?
Setting up a Linear Program

Introduce decision variables

\[ x_i = \text{credit in month } i \in \{1, \ldots, 6\} \]
\[ y_i = \text{amount of money due to commercial paper } i \in \{1, 2, 3\} \]
\[ z_i = \text{fund excess in month } i \in \{1, \ldots, 6\} \]

Define \( b = (-150, -100, 200, -200, 50, 300) \) as the vector of net cash flow. Then the following LP determines the optimal investment strategy:

\[
\begin{align*}
\text{max } & z_6 - x_6 \\
\text{subject to } & x_1 + y_1 - z_1 = -b_1 \\
& x_i + y_i - 1.01x_{i-1} + 1.003z_{i-1} - z_i = -b_i \quad \forall i = 2, 3 \\
& x_i - 1.02y_{i-3} - 1.01x_{i-1} + 1.003z_{i-1} - z_i = -b_i \quad \forall i = 4, \ldots, 6 \\
& 0 \leq x_i \leq 100 \quad \forall i = 1, \ldots, 6 \\
& z_i \geq 0 \quad \forall i = 1, \ldots, 6 \\
& y_i \geq 0 \quad \forall i = 1, 2, 3 
\end{align*}
\]
Notation

Let $A \in \mathbb{R}^{m \times n}$, $v \in \mathbb{R}^n$ and $J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}$ with $j_1 \leq j_2 \leq \ldots \leq j_k$.

- For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$:
  - $a^j$: $j$-th column of $A$
  - $a_i$: $i$-th row of $A$
  - $A(i, j)$: Element of $A$ in $i$-th row and $j$-th column
  - $v(i)$ (also $v_i$): $i$-th component of $v$

- $A_J$: Matrix $(a^{j_1}, \ldots, a^{j_k})$
- $v_J$: Vector $(v(j_1), \ldots, v(j_k))^T$

Example

- $A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 7 \end{pmatrix}$, $v^T = (1 \ 2 \ 3 \ 4)$, $J = \{2, 4\}$
- $a^3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$
- $A_J = \begin{pmatrix} 3 & 5 \\ 3 & 7 \end{pmatrix}$, $v_J = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
Definition: Linear Program

Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^{m} \), \( c \in \mathbb{R}^{n} \), \( I_{\geq}, I_{\leq}, I_{=} \subseteq \{1, \ldots, m\} \) and \( J_{\geq}, J_{\leq} \subseteq \{1, \ldots, n\} \)

Linear Program (LP) consists of:

- **Objective function:**
  \[
  \max c^{T}x \\
  \text{or } \min c^{T}x
  \]

- **Constraints**
  \[
  a_{i}^{T}x \geq b(i), \ i \in I_{\geq} \\
  a_{j}^{T}x \leq b(j), \ j \in I_{\leq} \\
  a_{k}^{T}x = b(k), \ k \in I_{=}
  \]

- **Bounds on Variables**
  \[
  x_{j_{\geq}} \geq 0 \\
  x_{j_{\leq}} \leq 0
  \]
Definitions

**feasible LP, bounded LP**

- $x^* \in \mathbb{R}^n$ is **feasible** of feasible solution of LP, if $x^*$ satisfies all constraints and bounds on variables
- LP is **feasible** if there exist feasible solutions of LP. Otherwise LP is **infeasible**
- LP is **bounded**, if LP is feasible and there exits constant $M \in \mathbb{R}$ with $c^T x < M$ for all feasible solutions $x \in \mathbb{R}^n$ if LP is a maximization problem or $c^T x > M$ for all feasible $x \in \mathbb{R}^n$ if LP is minimization problem
Transformations

1) \( \max c^T x \) can be replaced by \( \min -c^T x \)
2) \( a_i^T x \leq b(i) \) can be replaced by \( -a_i^T x \geq -b(i) \)
3) Bound \( x(j) \geq 0 \) or \( x(j) \leq 0 \) can be replaced by constraint \( e_j^T x \geq 0 \) or \( e_j^T x \leq 0 \) respectively, where \( e_j \) is \( j \)-th unit vector
4) \( a_i^T x \leq b(i) \) can be replaced by \( a_i^T x + s = b(i) \), where \( s \) is a new variable which is bounded by \( s \geq 0 \)
5) Conversion of an unbounded variable \( x(j) \) in a bounded variable: Introduce new bounded variables \( p_+(j), p_-(j) \geq 0 \) and replace each occurrence of \( x(j) \) by \( p_+(j) - p_-(j) \)
6) Conversion of a variable \( x(j) \) bounded by \( \leq 0 \) into a variable which is bounded by \( \geq 0 \): Replace bound \( x(j) \leq 0 \) by \( x(j) \geq 0 \) and replace each other occurrence of \( x(j) \) by \( -x(j) \) (also in objective function!)
Standard forms

- **Inequality standard form**: 
  \[
  \max c^T x \\
  Ax \leq b
  \]

- **Equality standard form or simply standard form**: 
  \[
  \min c^T x \\
  Ax = b \\
  x \geq 0
  \]
Example

\[ \begin{align*}
\max & 2 \cdot x(1) - 4 \cdot x(2) \\
3 \cdot x(1) + 2 \cdot x(2) & \geq 4 \\
2 \cdot x(1) + 4 \cdot x(2) & \geq 9 \\
x(2) & \leq 0
\end{align*} \]

is to be transformed into standard form.

Application of 1) and 2)

\[ \begin{align*}
\min & -2 \cdot x(1) + 4 \cdot x(2) \\
-3 \cdot x(1) - 2 \cdot x(2) & \leq -4 \\
-2 \cdot x(1) - 4 \cdot x(2) & \leq -9 \\
x(2) & \leq 0
\end{align*} \]
Application of 6)

\[
\begin{align*}
\min & -2 \cdot x(1) - 4 \cdot x(2) \\
& -3 \cdot x(1) + 2 \cdot x(2) \leq -4 \\
& -2 \cdot x(1) + 4 \cdot x(2) \leq -9 \\
& x(2) \geq 0
\end{align*}
\]

Application of 5)

\[
\begin{align*}
\min & -2 \cdot p_+ (1) + 2 \cdot p_- (1) - 4 \cdot x(2) \\
& -3 \cdot p_+ (1) + 3 \cdot p_- (1) + 2 \cdot x(2) \leq -4 \\
& -2 \cdot p_+ (1) + 2 \cdot p_- (1) + 4 \cdot x(2) \leq -9 \\
& p_+ (1), p_- (1), x(2) \geq 0
\end{align*}
\]

Application of 4)

\[
\begin{align*}
\min & -2 \cdot p_+ (1) + 2 \cdot p_- (1) - 4 \cdot x(2) + 0 \cdot s(1) + 0 \cdot s(2) \\
& -3 \cdot p_+ (1) + 3 \cdot p_- (1) + 2 \cdot x(2) + s(1) = -4 \\
& -2 \cdot p_+ (1) + 2 \cdot p_- (1) + 4 \cdot x(2) + s(2) = -9 \\
& p_+ (1), p_- (1), x(2), s(1), s(2) \geq 0
\end{align*}
\]
In the following, we will always assume LPs to be in standard form

\[
\begin{align*}
\min & \; c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\] (1)
Reminder

► Vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are **linearly independent**, if one has
  $\sum_{i=1}^{k} x(i) v_i \neq 0$ for all $x \neq 0 \mathbb{R}^k$

► **Column-rank** of $A$ is maximal number of linearly independent
  columns of $A$

► **Row-rank** of $A$ is maximal number of linearly independent
  rows of $A$

► Rank of $A$, rank($A$): Maximal number of linearly independent
  columns of rows of $A$

► $A$ has full row-rank (column-rank), if rank($A$) = $m$ (rank($A$) = $n$)
W.l.o.g. $A$ has full row-rank

- Suppose $\text{rank}(A) < m$ and let the first row be in the span of the other rows
  
  $$a_1 = \sum_{i=2}^{m} \lambda_i a_i$$
  with suitable numbers $\lambda_2, \ldots, \lambda_m \in \mathbb{R}$.

- If $\sum_{i=2}^{m} \lambda_i b(i) = b(1)$, then one has for all $x \in \mathbb{R}^n$ with $a_i^T x = b(i)$, $i = 2, \ldots, m$ also $a_1^T x = b(1)$ which means that the first equation in $Ax = b$ can be discarded.

- If $\sum_{i=2}^{m} \lambda_i b(i) \neq b(1)$, then there does not exist an $x \in \mathbb{R}^n$ with $Ax = b$ and the LP (1) is infeasible.
In the following we will assume that $A$ has full row-rank.
Lemma 2.1

If LP (1) has an optimal solution, then there exists an optimal solution \( x^* \in \mathbb{R}^n \) of (1) such that \( A_J \) has full column rank, where \( J = \{ j \mid x^*(j) > 0 \} \).

Intuition for proof of Lemma 2.1

Consider LP

\[
\begin{align*}
\min x_1 + x_2 \\
(1 \ 2 \ 3) x &= (6 \ 15) \\
x &\geq 0
\end{align*}
\]

\( x = (1, 1, 1)^T \) is feasible but columns of \( A \) corresponding to positive entries of \( x \) (matrix \( A \) itself) are not linearly independent.

Vector \( d = (1, -2, 1)^T \) satisfies \( Ad = 0 \).

\( A(x + 1/2d) = Ax = b \), \( x + 1/2d = (3/2, 0, 3/2) \) \( \Rightarrow \) \( x + 1/2d \) is feasible, with fewer nonzero entries and smaller objective value than \( x \).
Proof of Lemma 2.1

- Let \( x^* \) be an optimal solution and suppose that columns of \( A_J \) are linearly dependent
- Idea: Compute new optimal solution \( \tilde{x} \) with \( J' = \{ j | \tilde{x}(j) > 0 \} \subset J \)
- After finite number of repetitions of this step, one has optimal solution which satisfies the condition of the theorem
- Let \( \bar{J} = \{1, \ldots, n\} \setminus J \)
- \( A_J \) does not have full column rank \(\implies\) \( \exists d \in \mathbb{R}^n \) with \( d \neq 0 \), \( Ad = 0 \) and \( d(j) = 0 \) \( \forall j \in \bar{J} \)
- \(\implies x^* \pm \varepsilon d \) is feasible for \( \varepsilon > 0 \) sufficiently small.
Proof cont.

- \( c^T (x^* \pm \varepsilon d) = c^T x^* \pm \varepsilon c^T d \); since \( x^* \) is optimal it follows that \( c^T d = 0 \)

- Via eventually changing \( d \) to \(-d\) we can assume that there exists a \( j \in J \) with \( d(j) < 0 \)

- Consider \( x^* + \varepsilon d \). Goal: Choose \( \varepsilon > 0 \) in such a way that \( x^* + \varepsilon d \) still feasible (\( \iff x^* + \varepsilon d \geq 0 \)) but also \( (x^* + \varepsilon d)(j) = 0 \) for some \( j \in J \)

- How large can we choose \( \varepsilon > 0 \) without getting infeasible?

- Infeasibility can be caused by indices \( j \) with \( d(j) < 0 \); Let \( K \subseteq J \) be the set of these indices

- We need for all \( k \in K \) the condition:

\[
\begin{align*}
x^*(k) + \varepsilon d(k) & \geq 0 \\
\iff \varepsilon & \leq -x^*(k) / d(k)
\end{align*}
\]
Proof cont.

- Let $k' \in K$ be an index with

$$-x^*(k')/d(k') = \min_{k \in K} -x^*(k)/d(k)$$

and let $\varepsilon' = -x^*(k')/d(k')$

- Then $x^* + \varepsilon' d$ is feasible and $J' = \{j \mid (x^* + \varepsilon' d)(j) > 0\} \subset J$. □
## Definitions

<table>
<thead>
<tr>
<th>Basic solution, basic feasible solution, associated basis</th>
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<tbody>
<tr>
<td>▶ $x^* \in \mathbb{R}^n$ is called a <strong>basic solution</strong>, if $Ax^* = b$ and $\text{rank}(A_J) =</td>
</tr>
<tr>
<td>▶ <strong>Basis</strong> is index set $B \subseteq {1, \ldots, n}$ with $\text{rank}(A_B) = m$ and $</td>
</tr>
<tr>
<td>▶ $x^* \in \mathbb{R}^n$ with $A_Bx_B^* = b$ and $x^*(j) = 0$ for all $j \notin B$ is basic solution associated to basis $B$.</td>
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</tbody>
</table>
**Lemma 2.2**

*Each basic solution \( x^* \) is associated to (at least) one basis \( B \)*

**Proof**

- Let \( J = \{ j \mid x^*(j) \neq 0 \} \)
- Columns of \( A_J \) are linearly independent
- Augment \( J \) to index set \( B \supseteq J \) such that \( A_B \) is invertible
- One has \( A_B x_B^* = b \) and \( x^*(j) = 0 \) for all \( j \notin B \)

**Exercise**

*Show that a basic solution can be associated to two different bases.*
A naive algorithm for linear programming

Let \( \min\{c^T x \mid Ax = b, \ x \geq 0\} \) be a bounded LP

- Enumerate all bases \( B \subseteq \{1, \ldots, n\} \) \( O\left(\binom{n}{m}\right) = O(n^m) \) many
- Compute associated basic solution \( x^* \) with \( x_B^* = A_B^{-1} b \) and \( x_B^* = 0 \)
- Return the one which has smallest objective function value among the feasible basic solutions
- Running time \( O(n^m \cdot m^3) \)
- **Are there more efficient algorithms?**
Programming exercise (2P)

- Download the **boost-library** and install it (it is important that the directory “boost” that contains all the header files is in your include path; read installation notes!)
- Download all files from this directory
- Compile and run the file “toto.cc”.
- Now implement the naive algorithm for linear programming and apply it to find an optimal solution of the Cashflow Management problem from the beginning of this lecture. (You have to transform the LP into standard form first!)
- Send your code, together with detailed compile-instructions, to Thomas Rothvoss (thomas.rothvoss@epfl.ch)
- Due date: Sept 30, 2009, By then we must have received your code and we have been able to compile and run it. It is a good idea to send us your code before this date.
Summary of today's lecture

Primary objectives:

- Linear programming: Example, Definition ✔
- The standard form ✔
- Basic feasible solutions ✔
- Simplex algorithm
- Along the way: Develop your own toy-solver in C++ first steps
- Use of modeling languages and efficient LP-solvers
- Constructing dedicated bond-portfolio