Algorithms for Market Equilibria:

Given:
- \( B \): set of \( n \) buyers
- \( A \): set of \( n \) divisible goods
- \( e_i \): amount of money available to buyer \( i \)
- \( b_j \): \( j \) goods of type \( j \)
- \( u_{ij} \): utility of good \( j \) to buyer \( i \)

**Happiness of buyer:**

\[
happiness(i) := \sum_{j=1}^{n} u_{ij} x_{ij}.
\]

Optimal basket of goods for buyer \( i \):

Given by LP:

\[
\max \sum_{j=1}^{n} u_{ij} x_{ij}
\]

s.t. \( \sum_{j=1}^{n} p_j \cdot x_{ij} \leq e_i \)

\[
x_{ij} \leq b_j \quad \forall j=1,\ldots,n
\]

\[
x \geq 0
\]

Where \( p_j \) is price of good \( j \).
\( P_1, \ldots, P_n \) are market clearing prices. If, after each buyer buys his optimal basket of goods, then

- there are no goods left
- there is no deficiency of goods.

I.o.w: \( \forall j: \sum_{i=1}^{n'} x_{ij} = b_j \)

Goal of today's lecture: Show that market equilibria (market clearing) prices exist, and that these can be efficiently computed.

Need theory of Duality for (convex) programs

Consider: \( \min f_0(x) \) \hspace{1cm} (I)

s.t. \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)

with domain \( \mathcal{D} \subseteq \mathbb{R}^n \)

Lagrangian: \( L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \)

\( g(\lambda) = \inf_{x \in \mathcal{D}} L(x, \lambda) \)
Thm: (Weak Duality)

Suppose \( (E) \) has optimal solution with opt.
value \( p^* \), then \( \forall \lambda \geq 0: \ g(\lambda) \leq p^* \)

Proof: Suppose \( \bar{x} \) is feasible i.e. \( \bar{x} \in D \) and

\[
f_i(\bar{x}) \leq 0, i = 1, \ldots, m.
\]

Let \( \lambda \geq 0 \), then

\[
L(\bar{x}, \lambda) = f(\bar{x}) + \sum_{i=1}^{m} \lambda_i f_i(\bar{x}) \leq 0
\]

\[
\leq f(\bar{x}). \quad \text{This implies } g(\lambda) \leq p^*
\]

for each \( \lambda \geq 0 \).

Lagrange Dual Problem:

\[
\begin{align*}
\max & \quad g(x) \\
\text{s.t.} & \quad \lambda \geq 0
\end{align*}
\]
Observation:

\( g(\lambda) \) is concave

\( \theta \in [0, 1], \lambda_1, \lambda_2 \geq 0, x \in \mathcal{D} : \)

\[
L(x, \theta \cdot \lambda_1 + (1-\theta) \lambda_2) = f_0(x) + \theta \sum_{i=1}^{m} \lambda_i f_i(x) + (1-\theta) \sum_{i=1}^{m} \lambda_i f_i(x)
\]

\[
= \theta L(x, \lambda_1) + (1-\theta) L(x, \lambda_2)
\]

\[
\Rightarrow g(\theta \cdot \lambda_1 + (1-\theta) \cdot \lambda_2) \geq \theta g(\lambda_1) + (1-\theta) g(\lambda_2)
\]

\[
\Rightarrow \text{Finding } \max_{\lambda \geq 0} g(\lambda) \text{ is convex program,}
\]

regardless of whether (I) is convex program

Slater's condition: (baby version)

\[ \text{If } \mathcal{D} \text{ is full-dimensional and } f_0, f_1, \ldots, f_m \text{ are convex, then (I) satisfies Slater's condition.} \]

Remark: More general version can be found in book:

Conic Optimization; Boyd, Vandenberghe, p 226.
Thm: If (I) satisfies Slater's condition and (I) is feasible, then

\[ \max_{\lambda \geq 0} \ g(\lambda) = \min_{x \in X} f(x) = p^* \]

s.t. \( f_i(x) \leq 0 \)

Proof:

Define \( A = \{ (u, t) : \exists x \in D \text{ with } f_i(x) \leq u_i, i = 1, \ldots, m \}
\text{ and } f_0(x) \leq t \} \)

\( A \) is convex. Let \( x \in D \) satisfy \( f_i(x) < 0, i = 1, \ldots, m \)

Define \( B = \{ (0, s) : s < p^* \} \)

\( A \cap B = \emptyset \) since \( (0, s) \in B \) and \( (0, s) \in A \), then

\( \exists x \in D \text{ with } f_i(x) \leq 0, i = 1, \ldots, m \text{ and } f_0(x) \leq s < p^* \)

Separating hyperplane thm ⇒

\[ \exists (\lambda, \mu) \neq 0 \text{ and } d \in \mathbb{R} \text{ s.t.} \]

\( (u, t) \in A \Rightarrow \lambda^T u + \mu^T t \geq d \) \( (*) \)

\( (u, t) \in B \Rightarrow \lambda^T u + \mu^T t \leq d \) \( (** \text{ sim}) \)
\( \lambda > 0 \) and \( \mu > 0 \), otherwise \( \lambda \cdot \mu + \mu \cdot t \) is unbounded from below if evaluated at \( t \).

\((**): \) \( \mu \cdot t \leq \lambda \) \( \forall \ t < \rho^* \) and thus \( \mu \cdot \rho^* \leq \lambda \)

Together with (**): \( \forall x \in D: \)

\[
\sum_{i=1}^{m} \lambda_i \cdot f_i(x) + \mu \cdot f_0(x) \geq \lambda \geq \mu \cdot \rho^*
\]

if \( \mu < 0 \):

\[
\sum_{i=1}^{m} \frac{\lambda_i}{\mu} \cdot f_i(x) + \frac{f_0(x)}{\mu} \geq \rho^* \\
\Rightarrow g\left(\frac{1}{\mu}\right) \geq \rho^*.
\]

if \( \mu = 0 \):

\[
\sum_{i=1}^{m} \lambda_i \cdot f_i(x) \geq 0 \quad \forall x \in D.
\]

In particular for \( x \): \( \sum_{i=1}^{m} \lambda_i \cdot f_i(x) \geq 0 \)

\[
\Rightarrow x = 0 \quad \forall x.
\]
Suppose $f_i(x^*)$, $i=0, \ldots, m$ are differentiable but not necessarily convex.

Let $x^*$, $x^*$ be optimal primal and dual solutions with zero duality gap, i.e.

$x^*$ minimizes $L(x, x^*)$ over $x$. Thus gradient must vanish at $x^*$:

$$\nabla f_0(x^*) + \sum_{i=1}^{m} x^*_i \nabla f_i(x^*) = 0$$

Thus we have:

$$\begin{cases}
    f_i(x^*) \leq 0 & i=0, \ldots, m \\
    x^*_i \geq 0 & i=0, \ldots, m \\
    x^*_i f_i(x^*) = 0 & i=0, \ldots, m \\
    \nabla f_0(x^*) + \sum_{i=1}^{m} x^*_i \nabla f_i(x^*) = 0
\end{cases}$$

$\rightarrow$ KKT Conditions.
Back to Market Equilibria:

Eisenberg-Gale convex program:

Assume $b_i = 1 \forall i$ by scaling $u_{i,j}$.

$$\text{min} \quad \sum_{i=1}^{n} c_i \cdot \log \left( \sum_{j=1}^{n} u_{i,j} x_{i,j} \right)$$

s.t.

$$\sum_{i=1}^{n} x_{i,j} \leq 1 \quad \forall j \in A$$

$$x_{i,j} \geq 0 \quad \forall i \in B, \forall j \in B.$$ 

Satisfies Slater's condition.

KKT conditions $\mu_i := \sum_{j \in A} \text{var} \text{ of } \text{first constraint}$

set:

(i) $\forall j \in A: \mu_j \geq 0$

(ii) $\forall j \in A: \mu_j > 0 \Rightarrow \sum_{i \in A} x_{i,j} = 1$

(iii) $\forall i \in B, \forall j \in A: \frac{u_{i,j}}{\mu_j} \leq \sum_{j \in A} \frac{u_{i,j} x_{i,j}}{e_i}$

(iv) $\forall i \in B, \forall j \in A: x_{i,j} > 0 \Rightarrow \frac{u_{i,j}}{\mu_j} = \sum_{j \in A} \frac{u_{i,j} x_{i,j}}{e_i}$
Theorem: If each good has potential buyer, then equilibrium exists.

Proof:

\[ P_j \geq \frac{e_i u_{ij}}{\sum_{j} u_{ij} x_{ij}} > 0 \]  

by (iii): \[ \sum_{c \in A} x_{ij} = 1 \]

\[ \Rightarrow \text{all prices are strictly positive, all goods are sold.} \]

(iii) and (iv): If \( i \) gets good \( j \), then \( j \) is among goods that give buyer \( i \) max. utility per unit of money spent at current prices. Each buyer gets only a bundle consisting of her most desired goods, i.e. on optimal bundle.

(iv) equivalent to: \( \forall i \in B, \forall j \in \mathcal{A}: \quad \frac{e_i u_{ij} x_{ij}}{\sum_{j \in \mathcal{A}} u_{ij} x_{ij}} = P_j x_{ij} \)
Summing over all $j$:

$$e_i = \sum_{j \in \Theta} u_{ij} x_{ij} \leq \sum_{j \in \Theta} u_{ij} x_{ij} = \sum_{j} p_j x_{ij}$$

$$\Rightarrow \forall i \in B : e_i = \sum_{j} p_j x_{ij}$$

$$\Rightarrow \text{money is fully spent, maintain clusters.}$$