PART 5

DYNAMIC PROGRAMMING

Shortest paths in directed graphs

Directed graph

- Tuple $G = (V, A)$, where V is finite set of nodes and $A \subseteq V \times V$ is set of arcs (also called edges); arc from u to v is denoted by uv
- $c : A \longrightarrow \mathbb{R}$ weights on arcs
- Path is sequence $P = v_0, v_1, \ldots, v_k$ of nodes such that $v_{i-1} v_i \in A$ for $i = 1, \ldots, k$.
- Length of path $c(P) = \sum_{i=1}^{k} c(v_{i-1} v_i)$
- Path $v_0, v_1, \ldots, v_k$ with $v_0 = v_k$ is called cycle

Shortest path problem

Let $s \in V$. Determine for each $v \in V$ a shortest path from s to v.

Bellman's Equations

- Suppose G does not contain cycle of negative length
- Let $\ell(v)$ be length of shortest path from s to v (possibly ∞)
- Clearly $\ell(s) = 0$ (no neg. cycles!)
- For $uv \in A$ one has $\ell(v) \leqslant \ell(u) + c(uv)$
- If $v \neq s$, there must be a node u on shortest path from s to v immediately preceding v.
  For this u one has
  $$\ell(v) = \ell(u) + c(uv)$$
  (principle of optimality)
- The shortest path distances $\ell(v)$ satisfy Bellman's equations

$$\begin{aligned} x_s &= 0, \\ x_v &= \min\{x_u + c(uv) : uv \in A\}, \quad v \in V. \end{aligned}$$

Sufficiency

Theorem:

If G does not contain cycles of length $\leqslant 0$ and if one has a path from s to each other node v, then there is unique solution to Bellman's equations, where $x_u = \ell(u)$ for each $u \in V$.

Proof

- Let x be solution to Bellman's equations
- Let $v \in V$, $v \neq s$. There exists $uv \in A$ with $x_v = x_u + c(uv)$. Likewise, there exists $w \in V$ with $x_u = x_w + c(wu)$.
- If, repeating, this process, we come back to v again, we have found a cycle of length 0 which is excluded
- Consequently, we arrive at s and we have constructed a path $P = (s = v_0, v_1, \ldots, v_k = v)$ with $x_v = c(P) + x_s = c(P)$. This shows $x_v \geqslant \ell(v)$.
- Suppose $x \neq \ell$. Then there exists node v with $x_v > \ell(v)$ such that the node u preceding v on shortest path from s to v has $x_u = \ell(u)$.
- Bellman's equations imply $x_v \leqslant x_u + c(uv) = \ell(u) + c(uv) = \ell(v)$

Acyclic Graphs

- $G = (V, A)$ is acyclic if G does not contain cycles
- In this case, G has an ordering "$<$" of nodes, such that $uv \in A$ implies $u < v$ (depth first search)
- Assume nodes are set $\{1, \ldots, n\}$.

Bellman's equations acyclic graphs

$$\begin{aligned} x_1 &= 0 \\ x_j &= \min_{k<j}(x_k + c_{kj}) \quad j = 2, 3, \ldots, n. \end{aligned}$$

- Easily solved by substitution
- x_2 is determined from x_1, x_3 determined from x_1 and x_2 etc.
- Running time: $O(|A| + |V|)$

Bellman-Ford algorithm

Bellman-Ford algorithm

- x_v^j length of shortest path from s to v using at most j arcs
- Initialization: $x_s^1 = 0$, $x_v^1 = \begin{cases} c(sv) & \text{if } sv \in A \\ \infty & \text{otherwise.} \end{cases}$
- **for** $i = 2, \ldots, n$
  for $v \in V$
  $x_v^i = \min\{x_v^{i-1}, \min_{uv \in A}\{x_u^{i-1} + c(uv)\}\}$

Running time

- Iterations outer loop: $|V|$
- Each arc is considered exactly once in inner loop: $O(|A|)$
- Complexity: $O(|V| \cdot |A|)$.

Negative cycles

- If network does not contain negative cycles and v is reachable from s, then a shortest path uses at most $n-1$ arcs.
- Bellman-Ford is correct if no negative cycles are present.

Exercise

Design a polynomial-time algorithm which detects whether $G = (V, A)$ together with arc-weights c has a negative cycle.

Dynamic programming

A capital budgeting problem

4 million to be invested in projects in three different regions

At most one project can be run in each region

Project	Region 1 cost/profit	Region 2 cost/profit	Region 3 cost/profit
1	0/0	0/0	0/0
2	1/2	1/3	1/2
3	2/4	3/9	2/5
4	4/10	-	-

Enumeration

$4 \times 3 \times 3$ possibilities

Integer program

$x_{ij} = 1$ if project i is run in region j

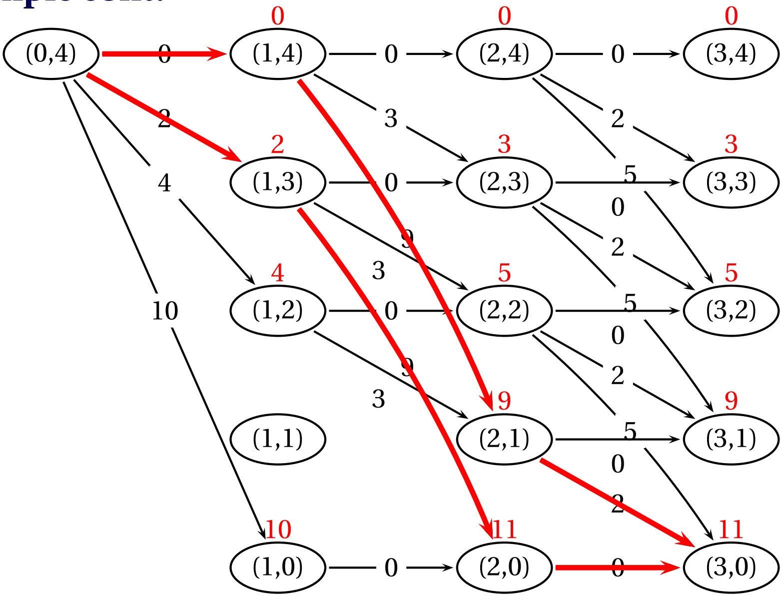
$$\begin{aligned}
&\max \textstyle\sum_{ij} p_{ij} x_{ij} \\
&\textstyle\sum_i x_{ij} \leqslant 1, \quad j = 1, \ldots, 3 \\
&\textstyle\sum_{ij} c_{ij} x_{ij} \leqslant 4 \\
&x_{ij} \in \{0, 1\}.
\end{aligned}$$

Dynamic program

States

- (i, j) represents optimal choice for first $i \in \{0, 1, 2, 3\}$ regions having still $j \in \{0, 1, 2, 3, 4\}$ mio. left to invest.
- Construct graph with nodes being states and arcs from stages (i, j) to $(i+1, j')$ having cost p if the investment of $j - j'$ reflects an implementation of a project of cost $j - j'$ and profit p in region $i+1$
- Longest path in this directed acyclic graph is optimal strategy
- Number of states: 4×5
- If we had $x \in \mathbb{N}$ regions with at least two possible projects per region and an amount of $y \in \mathbb{N}$ to invest, where each project would have integral costs, one would have $(x+1) \cdot (y+1)$ states, whereas enumeration would require at least 2^x decisions to be compared.

Example cont.



Forward and backward recursion

- Forward recursion: Label each state which is reachable from starting state with the longest path distance from starting state using profits as edge costs.
  Process starts from starting node
- Backward recursion: Label each state with largest profit that can be collected from this state.

The Knapsack Problem

Knapsack

- n items of weight $w_i \in \mathbb{N}$ and profit $p_i \in \mathbb{N}$, $i = 1, \ldots, n$
- Knapsack of capacity $K \in \mathbb{N}$
- Task: Compute subset of items which still fit into knapsack and maximizes the profit

Integer Program

$$\max \sum_{i=1}^{n} p_i x_i$$
$$\sum_{i=1}^{n} w_i x_i \leq K$$
$$x_i \in \{0, 1\}.$$

Dynamic program for Knapsack

- $W_i(p)$ least possible weight that has to be accumulated in order to have total profit p using only items from $\{1,\dots,i\}$
- $W_{i+1}(p) = \min\{W_i(p), W_i(p-p_{i+1}) + w_{i+1}\}$
- States: $\{(i,p) : i \in \{1,\dots,n\}, p \in \{0,\dots,n\cdot p_{\max}\}\}$, where $p_{\max} = \max\{p_i : i = 1,\dots,n\}$.
- There is an edge from (i,p) to $(i+1,p)$ of weight 0 and there is an edge from (i,p) to $(i+1,p+p_{i+1})$ of weight w_{i+1}.
- Compute shortest path distances from $(0,0)$ to all nodes in directed graph
- Number of nodes: $p_{\max}\cdot n^2$
- Number of arcs: $2\cdot p_{\max}\cdot n^2$

Theorem

The knapsack problem can be solved in time $O(n^2\cdot p_{\max})$.

Alternative dynamic program

- $P_i(w)$: Maximum profit which can be accumulated choosing items out of $\{1,\dots,i\}$ having total weight w.
- $P_{i+1}(w) = \max\{P_i(w), P_i(w - w_{i+1}) + p_{i+1}\}$
- States: $\{(i,w) : i \in \{1,\dots,n\}, w \in \{0,\dots,K\}\}$
- There is an edge from (i,w) to $(i+1,w)$ of weight 0 and an edge from (i,w) to $(i+1, w+w_{i+1})$ of weight p_{i+1}.
- Number of states: $n \cdot K$
- Number of arcs: $2 \cdot n \cdot K$
- Find longest paths from $(0,0)$ to other states
- Running time: $O(n \cdot K)$

Approximation algorithm

ε-approximation algorithm

- Let $\varepsilon > 0$ be a number
- The algorithm $\mathscr{A}$ computes an ε-approximation to the knapsack problem, if it computes a feasible solution x_ε with $p^T x_\varepsilon \geqslant (1-\varepsilon) p^T x_{OPT}$, where x_{OPT} is an optimal solution of the Knapsack problem
- $\mathscr{A}$ is fully polynomial time approximation scheme if the running time of $\mathscr{A}$ is polynomial in n and $1/\varepsilon$.

Designing $\mathscr{A}$

- Set last $t \geqslant 0$ bits of each p_i to 0 obtaining profits $\overline{p}_i$
- Let $x \in \{0,1\}^n$ be feasible knapsack solution.
- Clearly $p^T x \geqslant \overline{p}^T x \geqslant p^T x - n \cdot 2^t$
- Let x_1, x_2 be opt. sol w.r.t. p and $\overline{p}$ respectively
- One has

  $$p^T x_1 \geqslant p^T x_2 \geqslant \overline{p}^T x_2 \geqslant \overline{p}^T x_1 \geqslant p^T x_1 - n \cdot 2^t$$

  from which we conclude $p^T x_1 - p^T x_2 \leqslant n \cdot 2^t$
- We want $(p^T x_1 - p^T x_2)/p^T x_1 \leqslant \varepsilon$
- We have $(p^T x_1 - p^T x_2)/p^T x_1 \leqslant n \cdot 2^t / p_{\max}$
- If $n/p_{\max} > \varepsilon$ solve problem with exact algorithm which takes time $O(n^2 p_{\max}) = O(n^3/\varepsilon)$.
- Else find t with $\varepsilon/2 < n \cdot 2^t / p_{\max} \leqslant \varepsilon$
- Apply algorithm to instance defined by $\widetilde{p} = \overline{p}/2^t$.
- $\widetilde{p}_{\max} \leqslant 2^{-t} p_{\max} < 2 \frac{n \cdot 2^t}{\varepsilon} \cdot 2^{-t} = 2 \cdot n/\varepsilon$.
- Running time: $O(n^2 \widetilde{p}_{\max}) = O(n^3/\varepsilon)$

Fully polynomial time approximation scheme for Knapsack

Theorem

There exists an algorithm which, given a knapsack instance and $\varepsilon > 0$ computes a feasible solution x with

$$OPT(1-\varepsilon) \leqslant p^T x$$

Where OPT denotes the value of an optimal solution. The running time of the algorithm is $O(n^3/\varepsilon)$.

PART 5.1
OPTION PRICING

American call options

- Gives the holder right to purchase underlying security for prescribed amount strike price
- Valid until certain expiration date

Pricing derivative security

- S_0 current price of underlying security
- Two possible outcomes up and down at time 1:

$$S_1^u = S_0 \cdot u$$
$$S_1^d = S_0 \cdot d$$

- How to price the derivative security?

Replication

- Consider portfolio of $\triangle$ shares of the underlying and B cash
- Up-state: $\triangle \cdot S_0 \cdot u + B \cdot R$, ($R$ is risk-less interest rate)
- Down-state: $\triangle \cdot S_0 \cdot d + B \cdot R$
- For what values of $\triangle$ and B will portfolio have same payoff C_1^u and C_1^d of derivate?

$$\begin{aligned} \triangle \cdot S_0 \cdot u + B \cdot R &= C_1^u \\ \triangle \cdot S_0 \cdot d + B \cdot R &= C_1^d \end{aligned}$$

- One obtains

$$\triangle = \frac{C_1^u - C_1^d}{S_0(u-d)}$$

$$B = \frac{uC_1^d - dC_1^u}{R(u-d)}$$

- Since portfolio is worth $S_0\triangle + B$ today, this should also be price for derivate security

$$
\begin{aligned}
C_0 &= \frac{C_1^u - C_1^d}{u-d} + \frac{uC_1^d - dC_1^u}{R(u-d)} \\
&= \frac{1}{R}\left[\frac{R-d}{u-d}C_1^u + \frac{u-R}{u-d}C_1^d\right]
\end{aligned}
$$

Risk-neutral probabilities

$p_u = \frac{R-d}{u-d}$, $p_d = \frac{u-R}{u-d}$

Remark

There is arbitrage opportunity if $u > R > d$ is not satisfied.
Consequently $p_u, p_d > 0$ and since $p_u + p_d = 1$ one can interpret p_u and p_d as probabilities.

n time steps

- Basic period length (day or week …)
- Price of asset in period being S there is the "up" event (price uS) with probability p and "down" event (price dS) with probability $1 - p$
- Starting from price S_0 in period 0, in period k it is $u^j d^{k-j} S_0$ if there are j up- and $k - j$ down moves.
- Probability is $\binom{k}{j} p^j (1-p)^{k-j}$ (binomial distribution).

n time steps

Determining u, d and p

- S_k: Price in periods $k = 0, \dots, n$
- Assume to know : Mean value μ and volatility σ of $\ln(S_n/S_0)$
- Let $\triangle = 1/n$ be length between two consecutive periods
- Mean and volatility of $\ln(S_1/S_0)$ are $\mu\triangle$ and $\sigma\triangle$ respectively
- Direct computation yields $\mu\triangle = p \cdot \ln u + (1-p)\ln d$ and $\sigma^2\Delta = p \cdot (1-p)(\ln u - \ln d)^2$.
- Set $d = 1/u$ then equations simplify

$$\begin{aligned}\mu \cdot \triangle &= (2p-1)\ln u \\ \sigma^2 \cdot \triangle &= 4p(1-p)(\ln u)^2\end{aligned}$$

- Squaring first and adding it to second equation yields

$$(\ln u)^2 = \sigma^2\triangle + (\mu\triangle)^2.$$

Determining u, d and p cont.

-
$$\begin{aligned} u &= e^{\sqrt{\sigma^2\triangle+(\mu\triangle)^2}} \\ d &= e^{-\sqrt{\sigma^2\triangle+(\mu\triangle)^2}} \\ p &= \frac{1}{2}\left(1+\frac{1}{\sqrt{1+(\sigma^2/\mu^2)\triangle}}\right) \end{aligned}$$

- For small $\triangle$ this is approximately

$$\begin{aligned} u &= e^{\sigma\sqrt{\triangle}} \\ d &= e^{-\sigma\sqrt{\triangle}} \\ p &= \frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{\triangle}\right) \end{aligned}$$

Example

- 52 periods
- S_0 known and random price S_{52} with mean and standard deviation of $\ln(S_{52}/S_0)$ being 10% and 30% respectively. Since $\triangle = 1/52$ "is small" one has $u = e^{0.3/\sqrt{52}} = 1.0425$, $d = 0.9592$ and $p = 1/2 + (1 + \frac{0.1}{0.3 \cdot \sqrt{52}})$

Dynamic program to determine value of option

- Suppose strike price is c
- Work backwards from time N to time 0
- Nodes at time N are terminal nodes
- Option value $v(j,N)$ of terminal nodes at height j is

$$\max\{u^j d^{N-j} S_0 - c, 0\}$$

- Compute $v(k,j)$ from $v(k+1,j)$ and $v(k+1,j+1)$ using formula with risk-neutral probabilities $p_u = \frac{R-d}{u-d}$, $p_d = \frac{u-R}{u-d}$:

$$v(k,j) = \max\left\{\frac{1}{R}(p_u v(k+1,j+1) + p_d v(k+1,j)), u^j d^{k-j} S_0 - c\right\}$$

- Output $v(0,0)$

Example

Stock with

- Volatility of logarithm $\sigma = .2$
- Current price is 62
- What is price of American call option with expiration date in 5 months from now?
- Strike price c is 60
- Rate of interest is 10% compounded monthly

Compute values

- $\triangle = 1/12$, $u = 1.05943$, $d = .94390$, $R = 1 + 0.1/12 = 1.00833$
- $p_u = .55770$
- Fill table entries at the end of period. Example (upper right): $S_0 \cdot u^5 - c = 22.75$

Example cont.