Convexity

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Assignment Sheet 2

September 29, 2016

Exercise 1
Let $\mathrm{X} \subseteq \mathbb{R}^2$. For each point $x \in \mathrm{X}$, let us denote $V(x)$ the set of all points $y \in \mathrm{X}$ that can "see" x, i.e. points s.t. the segment xy is contained in X. More formally, for $x \in \mathrm{X}$ let

$$V(x) = \{y \in \mathrm{X} : \forall \lambda \in [0,1], \lambda x + (1-\lambda)y \in \mathrm{X}\}$$

The *kernel* of X is the set of all points $x \in \mathrm{X}$ for which $V(x) = \mathrm{X}$.

a) Prove that the kernel of any set $\mathrm{X} \subseteq \mathbb{R}^2$ is convex.

b) Construct a nonempty set $\mathrm{X} \subseteq \mathbb{R}^2$ such that each of its finite subsets can be seen from some point of X but the kernel of X is empty.

Exercise 2
Let E be the ellipsoid $f(B(0,1))$, where $f : \ x \mapsto Ax + b$ with a non-singular matrix $A \in \mathbb{R}^{n \times n}$.

1. Let $n = 2$ and
$$A = \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
Draw the ellipsoid E. What are the axes of E?

2. Let $\Lambda = \Lambda(B)$ be the lattice and $E = \{x \in \mathbb{R}^n \mid x^T Qx \leq 1\}$ with matrices
$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{pmatrix}.$$
Show that there is a bijection between the sets $\Lambda \cap B(0,1)$ and $E \cap \mathbb{Z}^2$.
[Hint: Is there a relation between the matrices B and Q?]

Exercise 3
Recall the definition of the successive minima of a (for this exercise) full-dimensional lattice $\Lambda \subseteq \mathbb{R}^n$.

$$\lambda_k := \min\{r \geq 0 \mid \dim(B(0,r) \cap \Lambda) \geq k\}, \qquad k = 1, \ldots, n.$$

This definition might suggest that any lattice Λ possesses a basis $B = (b_1, \ldots, b_n)$ with $\|b_k\| = \lambda_k$ for all k.

However, this is not true in general. Show for $n \geq 5$ that there exists a lattice where you cannot find a basis with this property.

Exercise 4 Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice of rank d.

1. Let $B \in \mathbb{R}^{n\times d}$ be a matrix whose columns are linearly independent vectors in Λ. Show that B is a basis of Λ if and only if the fundamental parallelepiped $\mathcal{P}(B) = \{Bt \mid t \in [0,1)^d\}$ associated to B does not contain any lattice point apart from 0.

2. Let $p \in \Lambda$ and $p \neq 0$. We call p *primitive* if for each $k \in \mathbb{N}_{\geq 2}$ the vector $\frac{1}{k}p \notin \Lambda$. Show that any primitive vector can be extended to a basis B of Λ.

Exercise 5 [$\star$] Prove that John's theorem achieves a better approximation ratio for centrally symmetric convex bodies, i.e. prove the following.

Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body. Show that there is an ellipsoid $\mathcal{E}$ (with the origin as center) s.t. $\mathcal{E} \subseteq K \subseteq C\sqrt{n}\mathcal{E}$ for some (large) constant $C > 0$.

The deadline for submitting solutions is **Thursday, October 06, 2016**.