Convexity

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Assignment Sheet 1
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Exercise 1
Let $A, B \subseteq \mathbb{R}^d$ be two convex sets.

1. Show $A + A = 2A$. Does this hold if A is not convex?
   (We understand the addition and multiplication of sets to be component-wise, i.e. $A + B = \{a + b \mid a \in A,\ b \in B\}$ and $2A = \{2a \mid a \in A\}$.)

2. Show that $A + B$ is convex.

3. Assume A and B are convex and closed. Show that $\text{conv}(A \cup B)$ is not necessarily closed. Can you give an additional condition to A and B such that $\text{conv}(A \cup B)$ is closed and prove it's sufficiency?

Exercise 2 [$\star$]

1. Let $K \subseteq \mathbb{R}^n$ be centrally symmetric, convex, compact and of positive volume. For any such K, define $\|\cdot\|_K : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ as
   $$\|x\|_K := \min\{r \geq 0 \mid x \in rK\}.$$
   Show that $\|\cdot\|_K$ is a norm.

2. Let $\|\cdot\|$ be any norm. Show that this norm is induced by a centrally symmetric, convex, compact set $K \subset \mathbb{R}^n$ of positive volume, i.e. there exists some K s.t.
   $$\|\cdot\| = \|\cdot\|_K.$$

Exercise 3
Prove the following variant of *Minkowski's theorem* (You may use Minkowski's theorem as seen in class):

Let $C \subseteq \mathbb{R}^d$ be symmetric around the origin, convex, closed and bounded, and suppose that $\text{vol}(C) \geq 2^d$. Then C contains at least one integer point different from 0.

Exercise 4
Give a proof of *Caratheodory's theorem*:
Let $\mathrm{X} \subseteq \mathbb{R}^d$. Then each point in $conv(\mathrm{X})$ is in $conv(\mathrm{S})$ for some $\mathrm{S} \subseteq \mathrm{X}$, $|\mathrm{S}| \leq d + 1$.

The deadline for submitting solutions is **Thursday, September 29, 2016**.