Convexity
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Assignment Sheet 1
September 22, 2016

Exercise 1
Let $A, B \subseteq \mathbb{R}^d$ be two convex sets.

1. Show $A + A = 2A$. Does this hold if $A$ is not convex?
   (We understand the addition and multiplication of sets to be component-wise, i.e. $A + B = \{a + b \mid a \in A, b \in B\}$ and $2A = \{2a \mid a \in A\}$.)

2. Show that $A + B$ is convex.

3. Assume $A$ and $B$ are convex and closed. Show that $\text{conv}(A \cup B)$ is not necessarily closed. Can you give an additional condition to $A$ and $B$ such that $\text{conv}(A \cup B)$ is closed and prove it’s sufficiency?

Exercise 2 [⋆]

1. Let $K \subseteq \mathbb{R}^n$ be centrally symmetric, convex, compact and of positive volume. For any such $K$, define $\| \cdot \|_K : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ as
   $$\| x \|_K := \min \{ r \geq 0 \mid x \in rK \}.$$  
   Show that $\| \cdot \|_K$ is a norm.

2. Let $\| \cdot \|$ be any norm. Show that this norm is induced by a centrally symmetric, convex, compact set $K \subset \mathbb{R}^n$ of positive volume, i.e. there exists some $K$ s.t.
   $$\| \cdot \| = \| \cdot \|_K.$$ 

Exercise 3
Prove the following variant of Minkowski’s theorem (You may use Minkowski’s theorem as seen in class):

Let $C \subseteq \mathbb{R}^d$ be symmetric around the origin, convex, closed and bounded, and suppose that $\text{vol}(C) \geq 2^d$. Then $C$ contains at least one integer point different from 0.

Exercise 4
Give a proof of Caratheodory’s theorem:

Let $X \subseteq \mathbb{R}^d$. Then each point in $\text{conv}(X)$ is in $\text{conv}(S)$ for some $S \subseteq X, |S| \leq d + 1$.

The deadline for submitting solutions is Thursday, September 29, 2016.