Convexity

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Assignment Sheet 3

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Exercise 1
Let $\Lambda \subset \mathbb{R}^d$ be a lattice, v a shortest non-zero vector in Λ and $(v, a_2, \ldots, a_d)$ a basis of Λ. Moreover, let pr be the projection map on the subspace of $\mathbb{R}^d$ orthogonal to v, i.e.

$$\begin{aligned} \mathrm{pr} : \mathbb{R}^d &\to \mathbb{R}^d \\ x &\mapsto \left(x - \frac{x^\intercal v}{v^\intercal v} v\right). \end{aligned}$$

Show that $\mathrm{pr}(\Lambda)$ is a $(d-1)$-dimensional lattice with basis $(\mathrm{pr}(a_2), \ldots, \mathrm{pr}(a_d))$, in other words,

$$\mathrm{pr}(\Lambda) = \{\mathrm{pr}(a_2)x_2 + \cdots + \mathrm{pr}(a_d)x_d \mid x_2, \ldots, x_d \in \mathbb{Z}\}.$$

Exercise 2
Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $\Lambda^\star \subset \mathbb{R}^d$ be its dual lattice. Show that the packing radii of Λ and $\Lambda^\star$ satisfy

$$\rho(\Lambda)\rho(\Lambda^\star) \leq \frac{d}{4}.$$

[Hint:] Use Minkowski's first theorem to bound the $\|\cdot\|_2$-length of a shortest vector.

Exercise 3
Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and $\Lambda^\star$ its dual. The lattice width of a convex body K w.r.t. Λ is defined as

$$w_\Lambda(K) := \min_{v \in \Lambda^\star \setminus \{0\}} \max_{x,y \in K} v^T(x-y).$$

Let $D \subset \mathbb{R}^d$ be a disc of radius R, i.e. $D = \{x \in \mathbb{R}^d : \|x - r\| \leq R\}$.

1. Show that, for $c \in \mathbb{R}^d$, we have

$$\max_{x \in D} c^\intercal x - \min_{x \in D} c^\intercal x = 2R\|c\|.$$

2. Conclude that $w_\Lambda(D) = 2\lambda_1^\star$, where $\lambda_1^\star$ is the length of a shortest non-zero dual lattice vector.

Exercise 4 Let $u_1, \ldots, u_d \in \Lambda$ be linearly independet lattice vectors. Prove that

$$\mu(\Lambda) \leq \frac{1}{2}\sum_{i=1}^{d} \|u_i\|.$$

Exercise 5 [$\star$] Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $x \in \mathbb{R}^d$ be a point. Prove that for every $v \in \Lambda^\star \setminus \{0\}$

$$\frac{\{\langle v, x\rangle\}}{\|v\|} \leq \mathrm{dist}(x, \Lambda),$$

where $\{r\} := |\lceil r \rfloor - r|$ is defined to be the distance from $r \in \mathbb{R}$ to the closest integer.

Exercise 6 Let $\theta_1, \ldots, \theta_n$ be real numbers such that $m_1\theta_1 + \cdots + m_n\theta_n + m_{n+1} = 0$ for integers $m_1, \ldots, m_{n+1}$ implies $m_1 = \cdots = m_{n+1} = 0$. Denote the distance from some real r to the closest integer by $\{r\} := |\lceil r \rfloor - r|$. Prove Kronecker's Theorem:

For any real vector $a = (\alpha_1, \ldots, \alpha_n)$ and for any $\epsilon > 0$ there exists a positive integer m such that $\{\alpha_i - m\theta_i\} < \epsilon$ for $i = 1, \ldots, n$.

[Hint:] Let $d = n + 1$ and let $\tau > 0$ be a number. Consider the set $\Lambda_\tau \subseteq \mathbb{R}^d$ of all integer linear combinations of the vectors $u_i = (e_i^T, 0)^T$, $i = 1, \ldots, n$ and $u_{n+1} = (\theta_1, \ldots, \theta_n, \tau^{-1})$. Convince yourself that Λ_τ is indeed a lattice and that the packing radius of the dual lattice grows to infinity when τ grows. Deduce that the covering radius of Λ_τ tends to 0 as τ grows. Conclude the theorem.

[Hint 2:] In order to show that the packing radius of the dual lattice grows to infinity, you can for example take a shortest non-zero vector $v \in \Lambda_\tau^\star$ and show that there exists τ' such that $\|w\| \geq 2\|v\|$ for all $w \in \Lambda_{\tau'}^\star \setminus \{0\}$.

The deadline for submitting solutions is **Thursday, October 20, 2016**.