Convexity

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Assignment Sheet 10

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Exercise 1 [$\star$]
Consider the following two statements.

(i) [The Brunn-Minkowski inequality.] For non-empty, compact $A, B \subseteq \mathbb{R}^n$,

$$\operatorname{vol}(A+B)^{1/n} \geq \operatorname{vol}(A)^{1/n} + \operatorname{vol}(B)^{1/n}$$

(ii) For all compact $C, D \subseteq \mathbb{R}^n$ and all $t \in (0,1)$,

$$\operatorname{vol}((1-t)C + tD) \geq \operatorname{vol}(C)^{1-t} \operatorname{vol}(D)^t.$$

1. Derive (ii) from (i); prove and use the inequality $(1-t)x + ty \geq x^{1-t}y^t$ positive reals $x, y \in \mathbb{R}$ and $t \in (0,1)$.
   [*Hint: For $t \in \{0,1\}$, the inequality is obviously true. Is there another value for t for which the inequality is easy to show? Can you go on from there?*]

2. Prove (i) from (ii).
   [*Hint: What is the problem with showing the reverse direction? What happens if two sets C', D' have the same volume? What can you do with C and D to get this property?*]

Exercise 2
Let $A, B \subseteq S^{n-1}$ be measurable sets with distance at least $2t$. Define a measure $\mu(C) = \operatorname{vol}(C) s_{n-1}^{-1}$, where $s_{n-1} = \operatorname{vol}_{n-1}(S^{n-1})$ is the volume of the sphere for any measurable set $C \subseteq S^{n-1}$.
Prove that $\min\{\mu(A), \mu(B)\} \leq 2e^{-t^2 n/4}$.

Exercise 3
Let E be an equator of the unit ball B_1^n and let A_t be a belt of width $2t$ around E for $t \in (0,1)$. Formally, $E = \{x \in S^{n-1} : a^\intercal x = 0\}$ and $A_t = \{x \in S^{n-1} : |a^\intercal x| \leq t\}$ for some $a \in \mathbb{R}^n \setminus \{0\}$.

Show that if $\Pr(A_t) = \frac{1}{2}$, then $t = O(n^{-\frac{1}{2}})$, that is, half of the measure on the sphere is concentrated in the strip of width $O(n^{-\frac{1}{2}})$ around an equator.

[*Hint:* Recall the measure concentration inequality on a sphere, that we saw on the lecture, for subset $X \in S^{n-1}$ with $\Pr(X) \geq \frac{1}{2}$. Apply this to the two halfspheres defined by the equator.]

The deadline for submitting solutions is **Thursday, December 15, 2016**.