Assignment Sheet 10  
December 8, 2016

Exercise 1 ⋆
Consider the following two statements.

(i) [The Brunn-Minkowski inequality.] For non-empty, compact $A, B \subseteq \mathbb{R}^n$,
$$\text{vol}(A + B)^{1/n} \geq \text{vol}(A)^{1/n} + \text{vol}(B)^{1/n}$$

(ii) For all compact $C, D \subseteq \mathbb{R}^n$ and all $t \in (0, 1)$,
$$\text{vol}((1 - t)C + tD) \geq (1 - t)^{1/n} \text{vol}(C) + t^{1/n} \text{vol}(D).$$

1. Derive (ii) from (i); prove and use the inequality
$$(1 - t)x + ty \geq (1 - t)x^1 + ty^1$$
for positive reals $x, y \in \mathbb{R}$ and $t \in (0, 1)$.

   [Hint: For $t \in \{0, 1\}$, the inequality is obviously true. Is there another value for $t$ for which the inequality is easy to show? Can you go on from there?]

2. Prove (i) from (ii).

   [Hint: What is the problem with showing the reverse direction? What happens if two sets $C', D'$ have the same volume? What can you do with $C$ and $D$ to get this property?]

Exercise 2
Let $A, B \subseteq S^{n-1}$ be measurable sets with distance at least $2t$. Define a measure
$$\mu(C) = \text{vol}(C) s_n^{-1},$$
where $s_n = \text{vol}_{n-1}(S^{n-1})$ is the volume of the sphere for any measurable set $C \subseteq S^{n-1}$.
Prove that $\min\{\mu(A), \mu(B)\} \leq 2e^{-t^2 n/4}$.

Exercise 3
Let $E$ be an equator of the unit ball $B^n$ and let $A_t$ be a belt of width $2t$ around $E$ for $t \in (0, 1)$. Formally,
$$E = \{x \in S^{n-1} : a^\top x = 0\} \text{ and } A_t = \{x \in S^{n-1} : |a^\top x| \leq t\} \text{ for some } a \in \mathbb{R}^n \setminus \{0\}.$$
Show that if $\Pr(A_t) = \frac{1}{2}$, then $t = O(n^{-\frac{1}{2}})$, that is, half of the measure on the sphere is concentrated in the strip of width $O(n^{-\frac{1}{2}})$ around an equator.

   [Hint: Recall the measure concentration inequality on a sphere, that we saw on the lecture, for subset $X \subseteq S^{n-1}$ with $\Pr(X) \geq \frac{1}{2}$. Apply this to the two halfospheres defined by the equator.]

The deadline for submitting solutions is Thursday, December 15, 2016.