Convexity

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Assignment Sheet 5

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Exercise 1
Let $K \subsetneq \mathbb{R}^n$ be a closed convex set and $p \in \mathbb{R}^n \setminus K$.

Prove that there exists a *unique* point $x \in K$ minimizing the distance to p, i.e. $\|x - p\| \leq \|y - p\|$ for all $y \in K$.

Exercise 2
Let $K \subset \mathbb{R}^d$ be a compact convex body with a non-empty interior and suppose you are given E_{in}, the ellipsoid of largest volume contained in K.
Show how to compute a vector $u \in \mathbb{Z}^d$ s.t. $\max_{x,y \in K} u^\intercal(x - y) \leq d \cdot w(K)$ by one shortest lattice vector computation, where $w(K)$ is defined to be

$$w(K) = \min_{u \in \mathbb{Z}^d \setminus \{0\}} \max_{x,y \in K} u^\intercal(x - y)$$

Exercise 3 [$\star$]
Two sets $X, Y \subseteq \mathbb{R}^n$ are called *strictly separable* if there is a hyperplane $a^T x = b$ such that $a^T x < b$ for all $x \in X$ and $a^T y > b$ for all $y \in Y$.
Prove that two disjoint closed balls $B(z_1, r_1), B(z_2, r_2) \subseteq \mathbb{R}^n$ are strictly separable.
Prove or disprove the following statement: Any two disjoint closed convex sets are strictly separable.

Exercise 4
Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice and $\mathcal{V}$ its voronoi cell.

1. Show $\operatorname{vol} \mathcal{V} = \det \Lambda$.
2. Show $\mu(\Lambda) = \max_{x \in \mathcal{V}} \|x\|$.

Exercise 5
Let C be a convex cone and $-C$ the cone $\{x : -x \in C\}$. We call $L = C \cap -C$ the *lineality space* of C. We call a cone *pointed* if 0 is an extreme point.

1. Prove that $\overline{C} := C \cap L^\perp$, where $L^\perp = \{u : u^T x = 0 \ \forall x \in L\}$, is a pointed cone and that C is the direct sum of its lineality space L and the pointed cone $\overline{C}$, i.e.

$$C = (C \cap L^\perp) \oplus L.$$

2. Show that any polyhedron has a decomposition

$$P = (Q + C) \oplus L,$$

where Q is a polytope, C is a pointed cone and L is a linear subspace.

[*Attention: in this exercise, $\oplus$ denotes the direct sum, while we refer to Minkowski's sum by $+$.*]

The deadline for submitting solutions is **Thursday, November 03, 2016**.