Convexity

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Assignment Sheet 4

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Exercise 1

Consider $\Lambda = \mathbb{Z}^n$. Show that there exists a convex body K not containing any integer point in its interior such that for each nonzero integral vector y one has

$$\max_{x \in K} y^T x - \min_{x \in K} y^T x \geq n.$$

(For simplicity, it might contain lattice points on its boundary.)

Exercise 2

Let $P \subseteq \mathbb{R}^n$ be a polyhedron given by the system $Ax \leq b$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

We call a point $p \in P$ a *vertex* of P if there exists an inequality $c^T x \leq \delta$ valid for P such that $\{x \in P : c^T x = \delta\} = \{p\}$.

We call a point $p \in P$ an *extreme point* of P if it cannot be written as the convex combination of other points in P, i.e. there exist no two points $q_1, q_2 \in P$ distinct from p and $\lambda \in (0, 1)$ with $p = \lambda q_1 + (1 - \lambda) q_2$. Show the following.

1. If $x^\star$ is a vertex of P, then $x^\star$ is an extreme point of P.
2. If there is a subsystem $A'x \leq b'$ of $Ax \leq b$ with $A'x^\star = b'$ and $\operatorname{rank}(A') = n$, then $x^\star$ is a vertex of P.
3. If $x^\star$ is an extreme point of P, then there is a subsystem $A'x \leq b'$ of $Ax \leq b$ with $A'x^\star = b'$ and $\operatorname{rank}(A') = n$.

Exercise 3

Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix and let $a_1, \ldots, a_n$ denote the columns of A. Show that $\operatorname{cone}(\{a_1, \ldots, a_n\})$ is the polyhedron $P = \{x \in \mathbb{R}^n : A^{-1} x \geq 0\}$.

Show that $\operatorname{cone}(\{a_1, \ldots, a_k\})$ for $k \leq n$ is the set

$$P_k = \{x \in \mathbb{R}^n : \; a_i^{-1} x \geq 0, \; i = 1, \ldots, k; \; a_i^{-1} x = 0, \; i = k+1, \ldots, n\},$$

where a_i^{-1} denotes the i-th row of A^{-1}.

Exercise 4 [$\star$]
Let $K \subseteq \mathbb{R}^n$ be a convex set. The *polar* of K is defined as

$$K^\star := \{y \in \mathbb{R}^n : \; y^T x \leq 1 \;\forall x \in K\}.$$

(You might want to draw the cube $[-1,1]^2$ *and its polar, and the unit ball and its polar to see what happens. This is not part of the exercise.)*
Show the following.

1. $K^\star$ is a closed convex set that contains the origin.
2. If K is bounded, then $0 \in \operatorname{int} K^\star$.
3. If $0 \in \operatorname{int} K$, then $K^\star$ is bounded.
4. $K_1 \subseteq K_2$ implies $K_1^\star \supseteq K_2^\star$.
5. $K \subseteq (K^\star)^\star$.
6. If K is closed and contains the origin, then $(K^\star)^\star = K$.

The deadline for submitting solutions is **Thursday, October 27, 2016**.