Strong Relaxations for Discrete Optimization Problems	8/04/16

Lecture Extended formulations: introduction and basic techniques

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1 Introduction

Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, an *extended formulation* for P is a system $Cx + Dy \leq d$, such that P=$\{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^p : Cx + Dy \leq d\}$. Denoting by $Q = \{(x, y) \in \mathbb{R}^{n+p} : Cx + Dy \leq d\}$, we will write $P = proj_x(Q)$.

Now let

$$Q = \{(x, y) : C^{=}x + D^{=}y = d^{=}; C^{<}x + D^{<}y \leq d^{<}\},$$

where $C^{=}x + D^{=}y = d^{=}$ is the set of implicit equations from $Cx + Dy \leq d$, and $C^{<}x + D^{<}y \leq d^{<}$ are the remaining inequalities. We define the *size* of Q to be the number of inequalities from the system $C^{<}x + D^{<}y \leq d$.

The *extension complexity* of a polyhedron P, denoted by $xc(P)$, is the minimum size of an extended formulation for P. In this lecture, we will see some classical techniques to obtain extended formulations – hence, to upper bound $xc(P)$ – and their applications. We are mostly interested in extended formulations of polynomial size – we will call those *compact*. We start with an easy example.

Example 1 *The* crosspolytope B_n *is the convex hull of* $\pm e_i$*, where* e_i *is the vector in* $\mathbb{R}^n$*, having* 1 *on the* i*-th position, and* 0 *everywhere else. One can show that*

$$B_n = \{x : x(I) - x([n]\backslash I) \leq 1, \text{ for } I \subseteq [n]\},$$

and the previous description is non-redundant. Hence, B_n *has* $\Theta(2^n)$ *facets. Now let* $v_1, \ldots, v_{2n}$ *be the vertices of* B_n*. Recall that a polytope is exactly the convex hull of its vertices. Let*

$$Q_n = \{(x, \lambda) : x = \sum_{i=1}^{2n} \lambda_i v_i, \sum_i \lambda_i = 1, \lambda \geq 0\}.$$

Clearly $proj_x(Q_n) = B_n$*, so* $xc(B_n) = \Theta(n)$*.*

We say that a polyhedron $Q \subseteq \mathbb{R}^p$ is an *extension* of a polyhedron P if there exists an affine map f, such that $P = f(Q)$. The *size* of an extension is the number of facets of Q. The following is an easy consequence of results in polyhedral theory we saw during previous lectures.

Exercise 1 *Let* P *be a polyhedron. Then* $xc(P)$ = *minimum size of* Q*, where* Q *is an extension of* P*.*

Let us remark that, with the technique seen in Example 1, we can always give an extension of a polytope that is a simplex. Moreover, because of Exercise 1, we will sometime abuse notation and speak of an extended formulation when referring to a polyhedron, instead of the system of inequalities defining it.

Lemma 1 *Let* P *be a polyhedron and* F *be a face of* P*. If* Q *is an extended formulation for* P*, then* $F = proj_x(F')$*, where* F' *is a face of* Q*.*

Proof Let F be a face whose associated defining inequality is $cx = \delta$. Then, the inequality $cx \leq \delta$ is valid both for P and Q. Therefore, $cx = \delta$ defines a face F' of Q. Thus

$$F = \{x : \exists y \text{ such that } (x, y) \in Q, cx = \delta\} = \{x : \exists y \text{ such that } (x, y) \in F'\} = proj_x(F'),$$

which completes the proof of the lemma. ■

2 Union of polytopes and all even subsets of a set

Let $P_n = conv\{\chi^S : S \subseteq [n], |S| \equiv 0 \mod 2\}$ be the convex hull of the characteristic vectors of all even subsets of a set. Using a primal-dual argument, one can show the following exponential-size description:

$$P_n = \{x : x(S) - x([n]\backslash S) \leq |S| - 1, \forall S \subseteq [n], |S| \text{ odd}\}.$$

On the other hand, each *slice* $P_n^k = conv\{\chi^S : S \subseteq [n], |S| = k\}$ of P_n has a description with a linear number of inequalities, since

$$P_n^k = \{x : 0 \leq x \leq 1, \sum_{i=1}^{n} x_i = k\},$$

as each vertex of the latter description is clearly integral.

Our goal will be to obtain a compact extended formulation for P_n from the descriptions of P_n^k. More generally, given polytopes $P_1, \cdots, P_k$, we want to give an extended formulation for $conv(\bigcup_{i=1}^k P_i)$. This is proved in the following theorem.

Theorem 2 (Union of polytopes) *Let $P_1, \cdots, P_k \subseteq \mathbb{R}^n$, where $P_i = \{x : A_i x \leq b_i, 0 \leq x \leq u_i\}$, and $P = conv(\bigcup_{i=1}^k P_i)$. The following is an extended formulation for P:*

$$Q = \{(\lambda, x, \{y_i\}_{1\leq i\leq k}) \in \mathbb{R}^{k+n+kn} : \begin{array}{rlcll} & x & = & \sum_{i=1}^k y_i, & \\ & A_i y_i & \leq & b_i \lambda_i & \textit{for } i \in [k], \\ 0 \leq & y_i & \leq & u_i \lambda_i & \textit{for } i \in [k], \\ & \sum_{i=1}^k \lambda_i & = & 1 \textit{ for } \lambda \geq 0\}. & \end{array}$$

Proof We start by proving that $proj_x(Q) \supseteq P$. Let $x \in P$, and we prove that there exist $\lambda, \{y_i\}_{1\leq i\leq k}$ such that $(\lambda, x, \{y_i\}_{1\leq i\leq k}) \in Q$. Since $x \in P$, we have $x = \sum_i \lambda_i z_i$, $z_i \in P_i$, and $\sum_i \lambda_i = 1, \lambda_i \geq 0$. It is easy to check that $A_i \lambda_i z_i \leq b_i \lambda_i$ and $0 \leq \lambda_i z_i \leq u_i \lambda_i$. Thus $(\lambda, x, \{\lambda_i z_i\}_i) \in Q$.

We now prove that $proj_x(Q) \subseteq P$. Let $z = (\lambda, x, \{y_i\}_{1\leq i\leq k}) \in Q$, and we prove that $x \in P$. We first assume that $\lambda \in \{0,1\}^k$. This implies that $\lambda_t = 1$ for exactly one index t, and $\lambda_i = 0$ otherwise. One easily checks that in this case $x = y_t \in P_t$.

Let $\lambda \notin \{0,1\}^k$. We will express z as convex combination of the points $\{\overline{z}^t\}_{1\leq t\leq k}$, $y_t \neq 0$, where:

- $\overline{z}^t = (\overline{\lambda}^t, \overline{x}^t, \{\overline{y}_i^t\}_{i\in[k]}) \in Q$, for all t, and
- $\overline{\lambda}^t \in \{0,1\}^k$.

$\overline{z}^t$ is defined as follows:

- $\lambda_i^t = 1$ for $i = t$, and 0 otherwise;
- $\overline{x}^t = \frac{y_t}{\lambda_t}$;
- $\overline{y}_i^t = \frac{y_t}{\lambda_t}$ for $i = t$, and 0 otherwise.

Cleary $\overline{\lambda}^t \in \{0,1\}^k$. Using the previous part, one easily checks that $\overline{z}^t \in Q$. Simple algebraic manipulations show $z = \sum_t \lambda_t \overline{z}^t$.■

From the previous theorem, we conclude $xc(P) = O(\sum_{i=1}^k xc(P_i))$, so $xc(P_n) = O(n^2)$.

3 Fourier elimination and a relaxation of the cut polytope

Let us first recall the Fourier elimination procedure. Let x be an n-dimensional vector of variables and y a single variable. Given a system S of inequalities of the form

$$\begin{array}{llll} a_i x & +y & \leq b_i & \text{for } 1 \leq i \leq t \\ a_i x & -y & \leq b_i & \text{for } t+1 \leq i \leq k, \\ a_i x & & \leq b_i & \text{for } k+1 \leq i \leq m. \end{array}$$

Consider the system S' given by

$$\begin{array}{llll} a_i x + a_j x & \leq & b_i + b_j & \text{for } 1 \leq i \leq t,\ t+1 \leq j \leq k \\ a_i x & \leq & b_i & \text{for all } k+1 \leq i \leq m. \end{array}$$

During a past lecture, we proved the following.

Theorem 3 (Fourier elimination) *Let $\bar{x} \in \mathbb{R}^n$. $\bar{x}$ is a feasible solution to S' if and only if there exists a scalar $\bar{y}$ such that $(\bar{x}, \bar{y})$ is a feasible solution to S.*

We will now use Fourier elimination to obtain a compact extended formulation for the following polytope: Given a graph G, let

$$P_G^{cut} = \{x \in \mathbb{R}^E : x(F) - x(C \backslash F) \leq |F| - 1, \forall \text{ cycle } C \text{ of } G, \text{ and } \forall F \subseteq C,\ |F| \text{ odd},\ 0 \leq x \leq 1\}.$$

Note that P_G^{cut} is a relaxation of the cut polytope of G, i.e. the convex hull of $\delta(S)$, for some $S \subseteq V$. In several relevant cases – for instance, when G is planar – P_G^{cut} coincides with the cut polytope.

Lemma 4 *Let $G = (V, E)$, and $G' = (V, E')$ be two graphs, with $E \subseteq E'$. Then $proj_E(P_{G'}^{cut}) = P_G^{cut}$.*

Proof It is easy to see that $proj_E(P_{G'}^{cut}) \subseteq P_G^{cut}$. Thus we only to prove the inverse direction. Note that it is enough to prove in the case when $E' = E \cup \{e\}$. $P_{G'}^{cut}$ is the set of inequalities defined by

(a) $x_{e'} \leq 1$

(b) $-x_{e'} \leq 0$

(c) $x_e + x(F') - x(C' \backslash (F' \cup \{e\})) \leq |F'|$, when C' s a cycle, $e \notin F' \subseteq C'$, $|F'|$ even.

(d) $-x_e + x(F'') - x(C'' \backslash F'') \leq |F''| - 1$, when $C'' \cup \{e\}$ is a cycle, $e \notin C''$. $F'' \subseteq C''$, $|F"|$ even.

(e) All the remaining inequalities of $P_{G'}^{cut}$ where $x_{e'}$ does not appear.

Project out $x_{e'}$ as to obtain the polytope Q. We want to show that $Q \supseteq P_G^{cut}$. Using Fourier elimination, we observe that the only inequalities from Q that are not trivially implied by inequalities of P_G^{cut} are those obtained summing inequalities (c) and (d). Each of those inequalities looks as follows:

$$x(F' \Delta F'') - x(C \Delta C' \backslash (F' \Delta F'')) + 2[x(F' \cap F'') - x(C' \cap C'' \backslash (F' \Delta F''))] \leq |F'| + |F''| - 1. \quad (1)$$

Setting $F = F' \Delta F''$ and $C = C' \Delta C''$ and using $x \geq 0$, we write the following inequality dominating (1):

$$x(F) - x(C \backslash F) + 2x(F' \cap F'') \leq 2|F' \cap F''| + |F' \Delta F''| - 1 = |F| + 2|F' \cap F''| - 1. \quad (2)$$

Note that $C = \bigcup_{i=1,\ldots,k} C_i$, where C_i is the union of edge-disjoint cycles. Also, for $i \in [k]$, let $F_i = F \cap C_i$. F is the symmetric different of a set of even and a set of odd cardinality, hence it has odd cardinality. Hence, $|F_i|$ is odd for at least an i, say wlog $i = 1$. Therefore, the following are valid inequalities for P_G^{cut}:

$x(F_1) - x(C_1 \backslash F_1) \leq |F_1| - 1,$

$x(F_i) \leq |F_i|$ for $i = 2, ..., k,$

$-x(C_i \backslash F_i) \leq 0$ for $i = 2, ..., k,$

$2x(F' \cap F'') \leq 2|F' \cap F''|.$

Summing up these inequalities, we obtain inequality (2), as required. ■

Lemma 5 *Let G be a graph, C a cycle of G, and e a chord of C. Then, all the inequalities from P_G^{cut} generated by C, F, for all $F \subseteq C$, $|F|$ odd, are redundant.*

Proof Let Q be the polytope P_G^{cut} to which we removed the inequality generated by C, F. We show this latter inequality is valid for Q. Let C_1,C_2 be cycles so that $C_1 \cup C_2 = C \cup \{e\}$ and $C_1 \cap C_2 = \{e\}$. Let $F_i = C_i \cap F$, for $i = 1, 2$. Exactly one of $|F_i|$ is odd, say wlog $|F_1|$. We have that

$$x(F_1) - x(C_1 \backslash F_1) \leq |F_1| - 1 \text{ and } x(F_2 \cup \{e\}) - x(C_2 \backslash (F_2 \cup \{e\})) \leq |F_2|$$

are valid inequalities for Q. Their sum is

$$x(F) - x(C) \leq |F_1| + |F_2| - 1 = |F| - 1,$$

completing the proof.■

We can then conclude that, for any graph G with n nodes,

$$xc(P_G^{cut}) \leq xc(P_{K_n}^{cut}) = O(n^3),$$

where K_n is the complete graph on n nodes, the inequality follows from Lemma 4 and the equality from Lemma 5.

4 Projection cone

Let $Q = \{(x, y) \in \mathbb{R}^{n+p} : Ax + By \leq d\}$ be a polyhedron and $P = proj_x(Q)$. We define the *projection cone* of Q onto x as $U_{Q,x} = \{u \geq 0 : uB = 0\}$. Since, for every vector u with $u \geq 0$, $uB = 0$, the inequality $uAx \leq ub$ is valid for Q, we have that $uAx \leq ub$ is also valid for P. The next theorem states that the converse also holds.

Theorem 6 *Let $Q = \{(x, y) : Ax + By \leq b\}$ be a polyhedron. Then, $proj_x(Q) = \{x : uAx \leq ub$, for all $u \in U_{Q,x}\}$.*

Proof Let $x \in \mathbb{R}^n$. We observe that x belongs to $proj_x(Q)$ if and only if the polyhedron $\{y : By \leq b - Ax\}$ is not empty. This means that x does not belong to $proj_x(Q)$ if and only if the system $By \leq b - Ax$ is not feasible. By Farkas's Lemma, the system $By \leq b - Ax$ is not feasible if and only if there exists a vector $u \geq 0$ with $uB = 0$ and $uAx > ub$. Thus, the inequality $uAx \leq ub$ is valid for $proj_x(Q)$, but it is violated by x.■

In the next exercise sheet, we will see how to use the projection one to obtain an extended formulation for the dominant of the convex hull of all cuts separating two given nodes in a digraph.

5 Notes

The presentation in Sections 2, 3, and 4 follows

M. Conforti, G. Cornuejols, and G. Zambelli. Extended formulations in combinatorial optimization. Annals OR 204(1), 97–143, 2013.

References to the original papers can be found there. Theorem 2 can be extended to deal with polyhedra.