1 Introduction

Given a polyhedron \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \), an extended formulation for \( P \) is a system \( Cx + Dy \leq d \), such that \( P = \{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^p : Cx + Dy \leq d \} \). Denoting by \( Q = \{ (x, y) \in \mathbb{R}^{n+p} : Cx + Dy \leq d \} \), we will write \( P = \text{proj}_x(Q) \).

Now let \( Q = \{ (x, y) : C^w x + D^w y = d^w ; C^c x + D^c y \leq d^c \} \), where \( C^w x + D^w y = d^w \) is the set of implicit equations from \( Cx + Dy \leq d \), and \( C^c x + D^c y \leq d^c \) are the remaining inequalities. We define the size of \( Q \) to be the number of inequalities from the system \( C^c x + D^c y \leq d \).

The extension complexity of a polyhedron \( P \), denoted by \( xc(P) \), is the minimum size of an extended formulation for \( P \). In this lecture, we will see some classical techniques to obtain extended formulations – hence, to upper bound \( xc(P) \) – and their applications. We are mostly interested in extended formulations of polynomial size – we will call those compact. We start with an easy example.

Example 1 The crosspolytope \( B_n \) is the convex hull of \( \pm e_i \), where \( e_i \) is the vector in \( \mathbb{R}^n \), having 1 on the \( i \)-th position, and 0 everywhere else. One can show that

\[
B_n = \{ x : x(I) - x([n] \setminus I) \leq 1, \text{ for } I \subseteq [n] \},
\]

and the previous description is non-redundant. Hence, \( B_n \) has \( \Theta(2^n) \) facets. Now let \( v_1, \ldots, v_{2n} \) be the vertices of \( B_n \). Recall that a polytope is exactly the convex hull of its vertices. Let

\[
Q_n = \{ (x, \lambda) : x = \sum_{i=1}^{2n} \lambda_i v_i, \sum_i \lambda_i = 1, \lambda \geq 0 \}.
\]

Clearly \( \text{proj}_x(Q_n) = B_n \), so \( xc(B_n) = \Theta(n) \).

We say that a polyhedron \( Q \subseteq \mathbb{R}^p \) is an extension of a polyhedron \( P \) if there exists an affine map \( f \), such that \( P = f(Q) \). The size of an extension is the number of facets of \( Q \). The following is an easy consequence of results in polyhedral theory we saw during previous lectures.

Exercise 1 Let \( P \) be a polyhedron. Then \( xc(P) = \text{minimum size of } Q, \text{ where } Q \text{ is an extension of } P. \)

Let us remark that, with the technique seen in Example 1, we can always give an extension of a polytope that is a simplex. Moreover, because of Exercise 1, we will sometime abuse notation and speak of an extended formulation when referring to a polyhedron, instead of the system of inequalities defining it.

Lemma 1 Let \( P \) be a polyhedron and \( F \) be a face of \( P \). If \( Q \) is an extended formulation for \( P \), then \( F = \text{proj}_x(F') \), where \( F' \) is a face of \( Q \).

Proof Let \( F \) be a face whose associated defining inequality is \( cx = \delta \). Then, the inequality \( cx \leq \delta \) is valid both for \( P \) and \( Q \). Therefore, \( cx = \delta \) defines a face \( F' \) of \( Q \). Thus

\[
F = \{ x : \exists y \text{ such that } (x, y) \in Q, cx = \delta \} = \{ x : \exists y \text{ such that } (x, y) \in F' \} = \text{proj}_x(F'),
\]

which completes the proof of the lemma. \( \blacksquare \)
2 Union of polytopes and all even subsets of a set

Let $P_n = \text{conv}\{\chi^S : S \subseteq [n], |S| \equiv 0 \mod 2\}$ be the convex hull of the characteristic vectors of all even subsets of a set. Using a primal-dual argument, one can show the following exponential-size description:

$$P_n = \{x : x(S) - x([n]\setminus S) \leq |S| - 1, \forall S \subseteq [n], |S| \text{ odd}\}.$$  

On the other hand, each slice $P_n^k = \text{conv}\{\chi^S : S \subseteq [n], |S| = k\}$ of $P_n$ has a description with a linear number of inequalities, since

$$P_n^k = \{x : 0 \leq x \leq 1, \sum_{i=1}^n x_i = k\},$$

as each vertex of the latter description is clearly integral.

Our goal will be to obtain a compact extended formulation for $P_n$ from the descriptions of $P_n^k$. More generally, given polytopes $P_1, \cdots, P_k$, we want to give an extended formulation for $\text{conv}(\bigcup_{i=1}^k P_i)$. This is proved in the following theorem.

**Theorem 2 (Union of polytopes)** Let $P_1, \cdots, P_k \subseteq \mathbb{R}^n$, where $P_1 = \{x : A_i x \leq b_i, 0 \leq x \leq u_i\}$, and $P = \text{conv}(\bigcup_{i=1}^k P_i)$. The following is an extended formulation for $P$:

$$Q = \{(\lambda, x, \{y_i\}_{1 \leq i \leq k}) \in \mathbb{R}^{k+n+kn} : x = \sum_{i=1}^k y_i, \quad A_i y_i \leq b_i \lambda_i \quad \text{for } i \in [k], \quad 0 \leq y_i \leq u_i \lambda_i \quad \text{for } i \in [k], \quad \sum_{i=1}^k \lambda_i = 1 \text{ for } \lambda \geq 0\}.$$  

**Proof** We start by proving that $\text{proj}_x(Q) \supseteq P$. Let $x \in P$, and we prove that there exist $\lambda, \{y_i\}_{1 \leq i \leq k}$ such that $(\lambda, x, \{y_i\}_{1 \leq i \leq k}) \in Q$. Since $x \in P$, we have $x = \sum \lambda_i z_i$, $z_i \in P_i$, and $\sum \lambda_i = 1, \lambda_i \geq 0$. It is easy to check that $A_i \lambda_i z_i \leq b_i \lambda_i$ and $0 \leq z_i \leq u_i \lambda_i$. Thus $(\lambda, x, \{\lambda_i z_i\}_{1 \leq i \leq k}) \in Q$.

We now prove that $\text{proj}_z(Q) \subseteq P$. Let $z = (\lambda, x, \{y_i\}_{1 \leq i \leq k}) \in Q$, and we prove that $x \in P$. We first assume that $\lambda \in \{0,1\}^k$. This implies that $\lambda_i = 1$ for exactly one index $t$, and $\lambda_i = 0$ otherwise. One easily checks that in this case $x = y_t \in P_t$.

Let $\lambda \notin \{0,1\}^k$. We will express $z$ as convex combination of the points $\{\pi^t\}_{1 \leq t \leq k}$, $y_t \neq 0$, where:

- $\pi^t = (\lambda^t, x^t, \{\pi^t_i\}_{1 \leq i \leq k}) \in Q$, for all $t$, and
- $\lambda^t \in \{0,1\}^k$.

$\pi^t$ is defined as follows:

- $\lambda^t_i = 1$ for $i = t$, and $0$ otherwise;
- $x^t = \frac{y_t}{\lambda^t}$;
- $\pi^t_i = \frac{y_t}{\lambda^t}$ for $i = t$, and $0$ otherwise.

Clearly $\lambda^t \in \{0,1\}^k$. Using the previous part, one easily checks that $\pi^t \in Q$. Simple algebraic manipulations show $z = \sum \lambda_i \pi^t$.

From the previous theorem, we conclude $xc(P) = O(\sum_{i=1}^k xc(P_i))$, so $xc(P_n) = O(n^2)$.  

3 Fourier elimination and a relaxation of the cut polytope

Let us first recall the Fourier elimination procedure. Let \( x \) be an \( n \)-dimensional vector of variables and \( y \) a single variable. Given a system \( S \) of inequalities of the form

\[
\begin{align*}
  a_i x + y & \leq b_i & \text{for } 1 \leq i \leq t \\
  a_i x - y & \leq b_i & \text{for } t + 1 \leq i \leq k, \\
  a_i x & \leq b_i & \text{for } k + 1 \leq i \leq m.
\end{align*}
\]

Consider the system \( S' \) given by

\[
\begin{align*}
  a_i x + a_j x & \leq b_i + b_j & \text{for } 1 \leq i \leq t, t + 1 \leq j \leq k \\
  a_i x & \leq b_i & \text{for all } k + 1 \leq i \leq m.
\end{align*}
\]

During a past lecture, we proved the following.

**Theorem 3 (Fourier elimination)** Let \( \bar{x} \in \mathbb{R}^n \). \( \bar{x} \) is a feasible solution to \( S' \) if and only if there exists a scalar \( \bar{y} \) such that \((\bar{x}, \bar{y})\) is a feasible solution to \( S \).

We will now use Fourier elimination to obtain a compact extended formulation for the following polytope: Given a graph \( G \), let

\[
P^\text{cut}_G = \{ x \in \mathbb{R}^E : x(F) - x(C \setminus F) \leq |F| - 1, \forall \text{ cycle } C \text{ of } G, \text{ and } \forall F \subseteq C, \ |F| \text{ odd}, 0 \leq x \leq 1 \}.
\]

Note that \( P^\text{cut}_G \) is a relaxation of the cut polytope of \( G \), i.e. the convex hull of \( \delta(S) \), for some \( S \subseteq V \). In several relevant cases – for instance, when \( G \) is planar – \( P^\text{cut}_G \) coincides with the cut polytope.

**Lemma 4** Let \( G = (V, E) \), and \( G' = (V, E') \) be two graphs, with \( E \subseteq E' \). Then \( \text{proj}_E(P^\text{cut}_G) = P^\text{cut}_{G'} \).

**Proof** It is easy to see that \( \text{proj}_E(P^\text{cut}_G) \subseteq P^\text{cut}_{G'} \). Thus we only to prove the inverse direction. Note that it is enough to prove in the case when \( E' = E \cup \{e\} \). \( P^\text{cut}_{G'} \) is the set of inequalities defined by

\[
\begin{align*}
  (a) & \quad x_{e'} \leq 1 \\
  (b) & \quad -x_{e'} \leq 0 \\
  (c) & \quad x_e + x(F') - x(C' \setminus (F' \cup \{e\})) \leq |F'|, \text{ when } C' \text{ s a cycle, } e \notin F' \subseteq C', \ |F'| \text{ even.} \\
  (d) & \quad -x_e + x(F'') - x(C'' \setminus F'') \leq |F''| - 1, \text{ when } C'' \cup \{e\} \text{ is a cycle, } e \notin C''. \ F'' \subseteq C'', \ |F''| \text{ even.} \\
  (e) & \quad \text{All the remaining inequalities of } P^\text{cut}_{G'} \text{ where } x_{e'} \text{ does not appear.}
\end{align*}
\]

Project out \( x_{e'} \) as to obtain the polytope \( Q \). We want to show that \( Q \supseteq P^\text{cut}_{G'} \). Using Fourier elimination, we observe that the only inequalities from \( Q \) that are not trivially implied by inequalities of \( P^\text{cut}_{G'} \) are those obtained summing inequalities (c) and (d). Each of those inequalities looks as follows:

\[
x(F' \Delta F'') - x(C' \Delta C'' \setminus (F' \Delta F'')) + 2[x(F' \cap F'') - x(C' \cap C'' \setminus (F' \Delta F''))] \leq |F'| + |F''| - 1. \tag{1}
\]

Setting \( F = F' \Delta F'' \) and \( C = C' \Delta C'' \) and using \( x \geq 0 \), we write the following inequality dominating (1):

\[
x(F) - x(C \setminus F) + 2x(F' \cap F'') \leq 2|F' \cap F''| + |F' \Delta F''| - 1 = |F| + 2|F' \cap F''| - 1. \tag{2}
\]

Note that \( C = \bigcup_{i=1}^k C_i \), where \( C_i \) is the union of edge-disjoint cycles. Also, for \( i \in [k] \), let \( F_i = F \cap C_i \). \( F \) is the symmetric differer of a set of even and a set of odd cardinality, hence it has odd cardinality. Hence, \( |F_i| \) is odd for at least an \( i \), say \( \text{wlog } i = 1 \). Therefore, the following are valid inequalities for \( P^\text{cut}_{G'} \):
\[ x(F_1) - x(C_1 \setminus F_1) \leq |F_1| - 1, \]
\[ x(F_i) \leq |F_i| \text{ for } i = 2, \ldots, k, \]
\[ -x(C_i \setminus F_i) \leq 0 \text{ for } i = 2, \ldots, k, \]
\[ 2x(F' \cap F'') \leq |F' \cap F''|. \]

Summing up these inequalities, we obtain inequality (2), as required. \[ \square \]

**Lemma 5** Let \( G \) be a graph, \( C \) a cycle of \( G \), and \( e \) a chord of \( C \). Then, all the inequalities from \( P_G^{\text{cut}} \) generated by \( C, F \), for all \( F \subseteq C, |F| \text{ odd}, \) are redundant.

**Proof** Let \( Q \) be the polytope \( P_G^{\text{cut}} \) to which we removed the inequality generated by \( C, F \). We show this latter inequality is valid for \( Q \). Let \( C_1, C_2 \) be cycles so that \( C_1 \cup C_2 = C \cup \{e\} \) and \( C_1 \cap C_2 = \{e\} \). Let \( F_i = C_i \cap F \), for \( i = 1, 2 \). Exactly one of \( |F_i| \) is odd, say wlog \( |F_1| \). We have that
\[ x(F_1) - x(C_1 \setminus F_1) \leq |F_1| - 1 \text{ and } x(F_2 \cup \{e\}) - x(C_2 \setminus (F_2 \cup \{e\})) \leq |F_2| \]
are valid inequalities for \( Q \). Their sum is
\[ x(F) - x(C) \leq |F_1| + |F_2| - 1 = |F| - 1, \]
completing the proof. \[ \square \]

We can then conclude that, for any graph \( G \) with \( n \) nodes,
\[ xc(P_G^{\text{cut}}) \leq xc(P_{K_n}^{\text{cut}}) = O(n^3), \]
where \( K_n \) is the complete graph on \( n \) nodes, the inequality follows from Lemma 4 and the equality from Lemma 5.

### 4 Projection cone

Let \( Q = \{(x, y) \in \mathbb{R}^{n+p} : Ax + By \leq d\} \) be a polyhedron and \( P = \text{proj}_x(Q) \). We define the *projection cone* of \( Q \) onto \( x \) as \( U_{Q,x} = \{u \geq 0 : uB = 0\} \). Since, for every vector \( u \) with \( u \geq 0, uB = 0 \), the inequality \( uAx \leq ub \) is valid for \( Q \), we have that \( uAx \leq ub \) is also valid for \( P \). The next theorem states that the converse also holds.

**Theorem 6** Let \( Q = \{(x, y) : Ax + By \leq b\} \) be a polyhedron. Then, \( \text{proj}_x(Q) = \{x : uAx \leq ub, \text{ for all } u \in U_{Q,x}\} \).

**Proof** Let \( x \in \mathbb{R}^n \). We observe that \( x \) belongs to \( \text{proj}_x(Q) \) if and only if the polyhedron \( \{y : By \leq b-Ax\} \) is not empty. This means that \( x \) does not belong to \( \text{proj}_x(Q) \) if and only if the system \( By \leq b-Ax \) is not feasible. By Farkas’s Lemma, the system \( By \leq b-Ax \) is not feasible if and only if there exists a vector \( u \geq 0 \) with \( uB = 0 \) and \( uAx > ub \). Thus, the inequality \( uAx \leq ub \) is valid for \( \text{proj}_x(Q) \), but it is violated by \( x \). \[ \square \]

In the next exercise sheet, we will see how to use the projection one to obtain an extended formulation for the dominant of the convex hull of all cuts separating two given nodes in a digraph.

### 5 Notes

The presentation in Sections 2, 3, and 4 follows


References to the original papers can be found there. Theorem 2 can be extended to deal with polyhedra.