Slack matrices, nonnegative rank and extension complexity

Semester Project

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In this project, we will implement algorithms to compute the nonnegative rank of a matrix and use these algorithms to study the extension complexity of some 2-level polytopes, which is a mysterious open problem. As the setting is slightly technical, in the following we explain the main concepts we will deal with.

Nonnegative rank

The nonnegative rank of a matrix \( M \in \mathbb{R}^{m \times n} \geq 0 \) is the smallest \( r \) such that \( M = PQ \), \( P \in \mathbb{R}^{m \times r} \geq 0 \), \( Q \in \mathbb{R}^{r \times n} \geq 0 \). In other words, the goal is to factorize a (nonnegative) matrix into two nonnegative factors whose intermediate dimension is minimum: if we forget about nonnegativity, we obtain the usual definition of rank of a matrix. The nonnegative rank is a fundamental parameter that has many applications, for instance in machine learning, where the factorization needs to be nonnegative in order to be interpreted as a probability distribution.

Extension complexity and slack matrices

To get an idea of what extension complexity is about, consider the following example. Assume that we want to find a maximum weight spanning tree in a graph \( G \). There are very efficient algorithms for this task, but if we have further constraints in our problem (say we require our tree to have a particular structure) the algorithms can hardly be adapted and usually the only way is to use linear programming. In particular we want to optimize a linear objective function over the spanning tree polytope \( STP(G) \) of \( G \) (to which we would add our extra constraints). But there is a problem: if \( G \) has \( n \) vertices, \( STP(G) \) has a description with \( \Omega(2^n) \) inequalities, which makes linear programming impractical. However, by adding new variables, one can obtain a polytope \( Q \) whose description has only \( O(n^3) \) inequalities (see [1]) for more details) and such that, when projecting \( Q \) to the original space, we get \( STP(G) \) again. We can then perform linear optimization on \( Q \) efficiently and project back the solution on \( STP(G) \) to solve our problem. \( Q \) is called an extension of \( STP(G) \). Given a polytope \( P \in \mathbb{R}^n \), its extension complexity is the smallest number of facets of a polytope \( Q \) that is an extension of \( P \).

There is a surprising link between nonnegative rank and extension complexity, which goes through the concept of slack matrices. Given a polytope \( P \) for which we are given two descriptions, one in terms of inequalities \( (P = \{ x \in \mathbb{R}^n : a_i x \leq b_i, i = 1, \ldots, M \}) \) and one in terms of vertices \( (P = \text{conv}\{v_1, \ldots, v_N\}) \), the slack matrix \( S(P) \in \mathbb{R}^{M \times N} \) whose entry \( i, j \) represents the slack of the \( j \)-th vertex with respect to the \( i \)-th inequality, i.e. \( S_{i,j} = b_i - a_i v_j \).

**Theorem [1]** Given a polytope \( P \) and its slack matrix \( S(P) \), the extension complexity of \( P \) is equal to the nonnegative rank of \( S(P) \).
2-level polytopes

A polytope $P$ is called 2-level if, for any supporting hyperplane $H$ defining the facet $F$, there is a hyperplane parallel to $H$ that contains all the vertices of $P$ that are not in $F$. The cube and the simplex are examples of 2-level polytopes. It is not hard to see (try!) that a polytope is 2-level if and only if it has a slack matrix with only 0/1 entries. The 2-level property appears in a variety of combinatorial contexts and the simple structure of such polytopes makes them interesting objects from the point of view of optimization. However, our knowledge about them is still relatively poor and there are many important open questions, in particular with respect to their extension complexity: for a 2-level polytope $P$ of dimension $d$, the only bounds known for its extension complexity $xc(P)$ are $d \leq xc(P) \leq 2^d$, and it is a challenging problem to improve these bounds. In fact, the only result in this direction is a class of polytopes with $xc(P) \geq d \log d$ (see [2]), so we don’t even know of any 2-level polytope with, say, quadratic extension complexity. The result of [2], as well as many other results on extension complexity, is obtained through some non-trivial techniques that exploit properties of $P$ and its slack matrix to bound $xc(P)$. However, for polytopes of small dimension, one could also compute the nonnegative rank exactly and use the previous theorem to obtain $xc(P)$. This would offer new insights as to which 2-level polytopes are more interesting for proving lower bounds on the extension complexity.

Aim of the project

In this project, the student will study some of the existing literature on the topic and implement algorithms to compute the nonnegative rank (from [3, 4]). Then, the student will perform extensive computations on certain 0/1 slack matrices (for instance, coming from stable set polytopes, see [2]) in order to find promising examples of 2-level polytopes with high extension complexity. If the research is successful, this might lead to the discovery of new interesting classes of 2-level polytopes.

Prerequisites

Advanced programming skills, and a good knowledge of discrete optimization.

If you are interested please do not hesitate to contact me at: manuel.aprile@epfl.ch

References


